Behavioural Metrics

Compositionality of the Kantorovich Lifting and an Application to Up-To Techniques

**Keri D'Angelo (Cornell), Sebastian Gurke (Duisburg-Essen), Johanna Maria Kirss (Copenhagen),
Barbara König (Duisburg-Essen), Matina Najafi (Amirkabir), Wojciech Różowski (UCL), Paul Wild (Erlangen-Nürnberg)**


TCO21

We look at linear-time behaviours of transition systems

We look at linear-time behaviours of transition systems

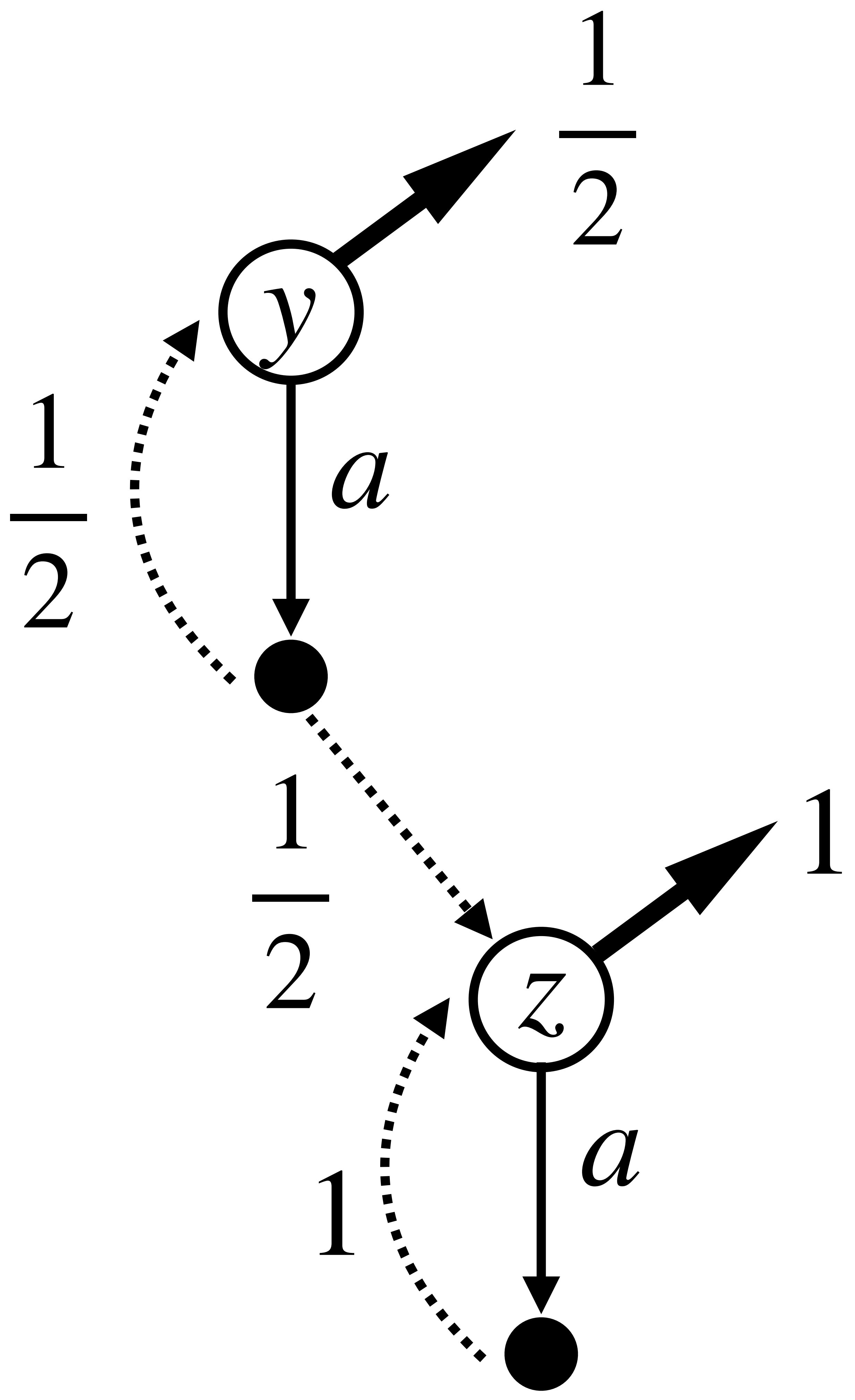
Instead of exact equivalence, we check **how far apart two states are**

We look at linear-time behaviours of transition systems

Instead of exact equivalence, we check **how far apart two states** are

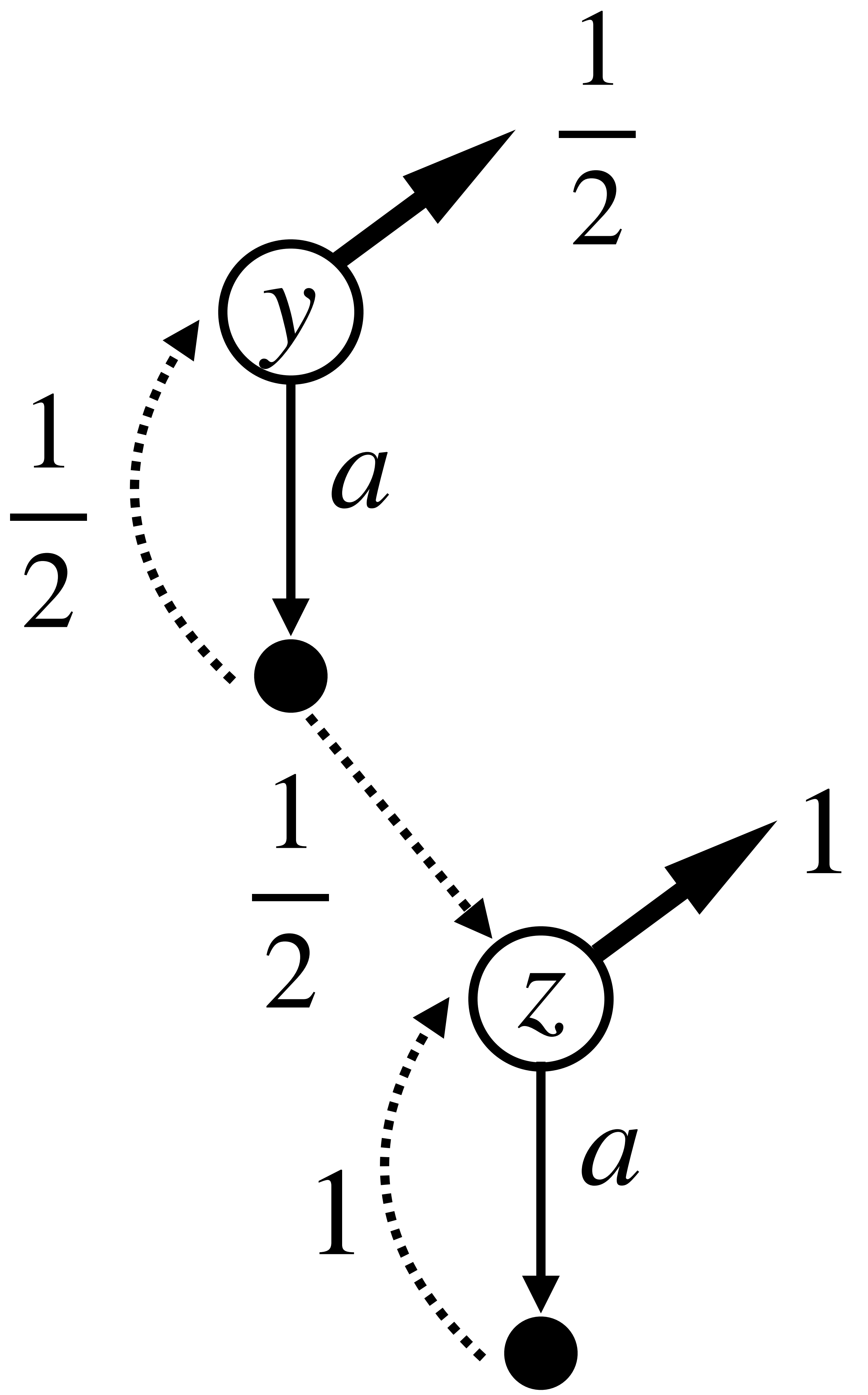
We give a **sound** technique for checking bounds on these distances

Rabin probabilistic automata



$$(X, X \to [0,1] \times \mathscr{D}(X)^A)$$

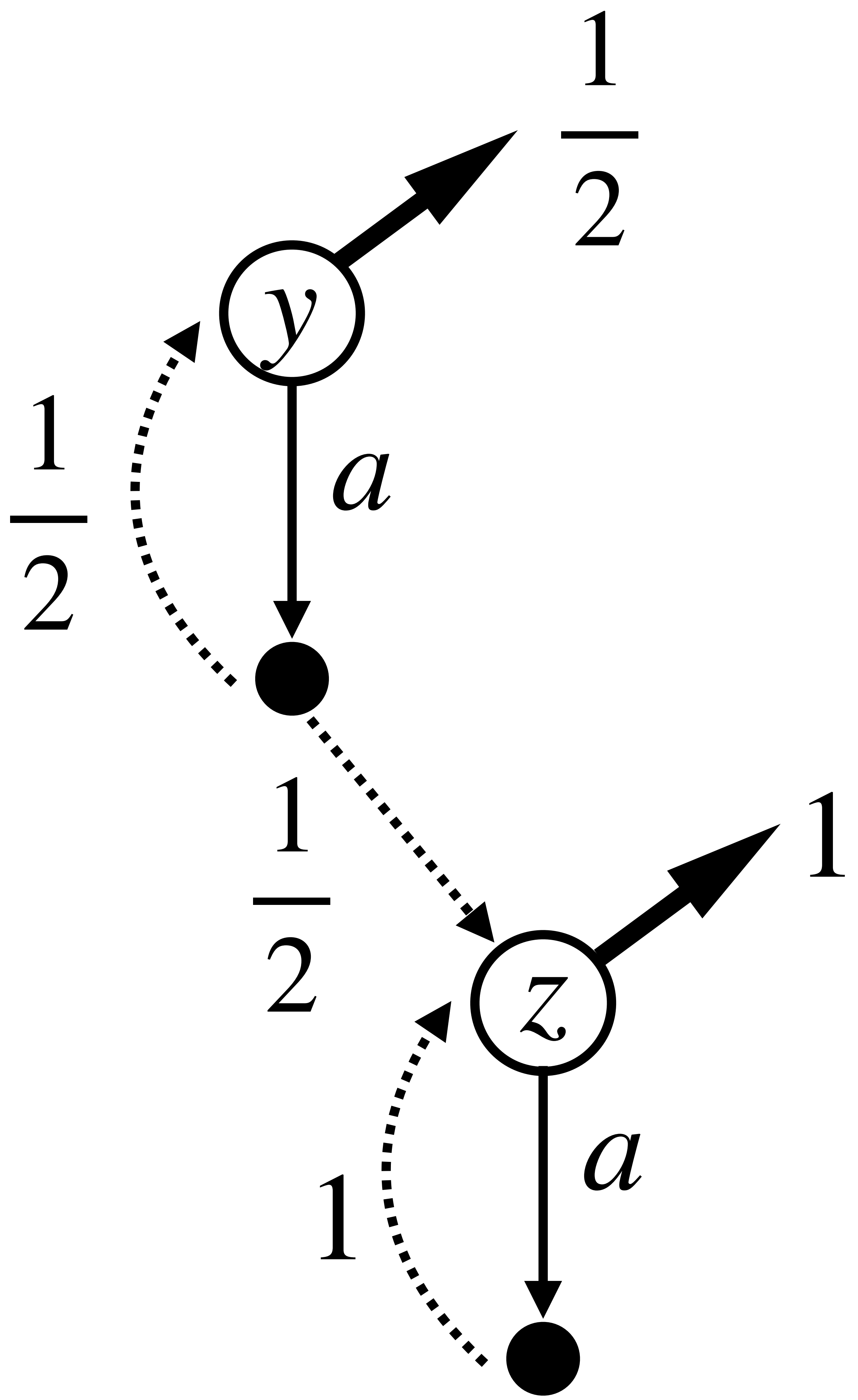
Rabin probabilistic automata

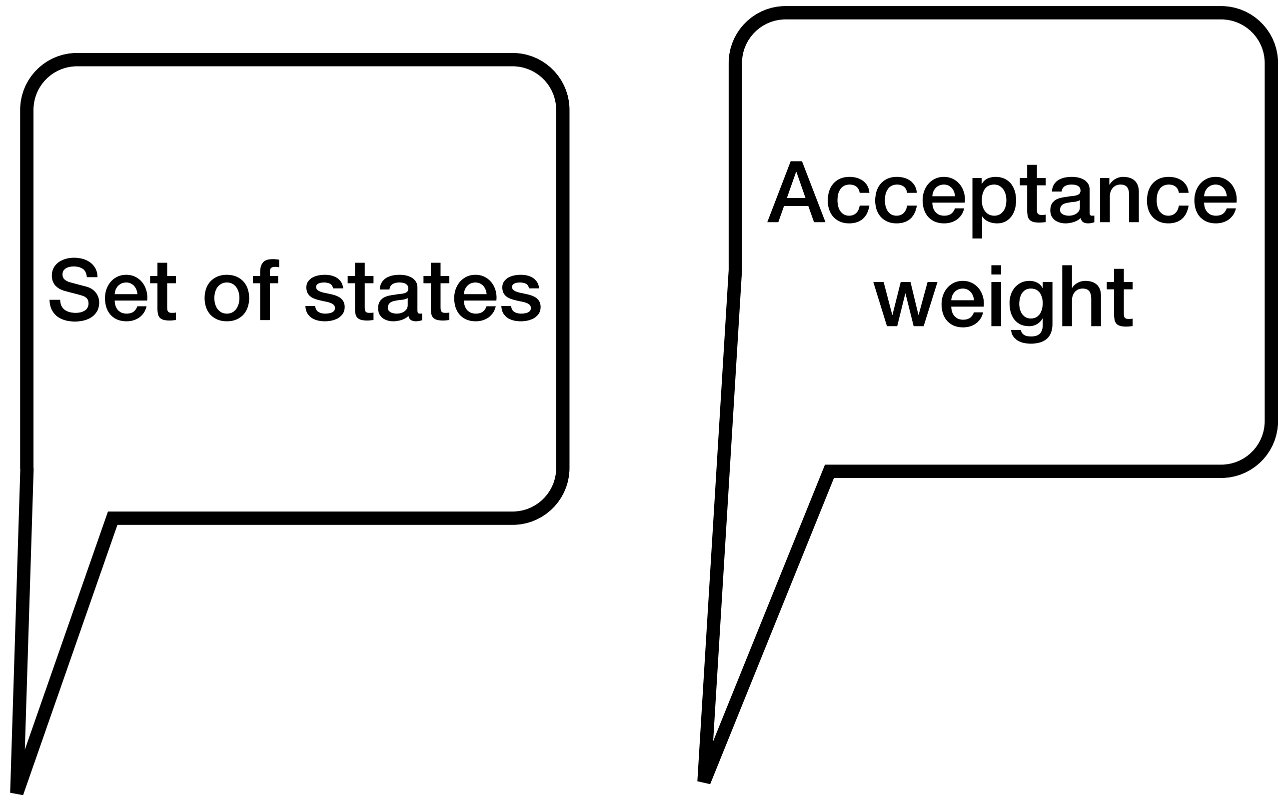


Set of states

$$(X, X \to [0,1] \times \mathscr{D}(X)^A)$$

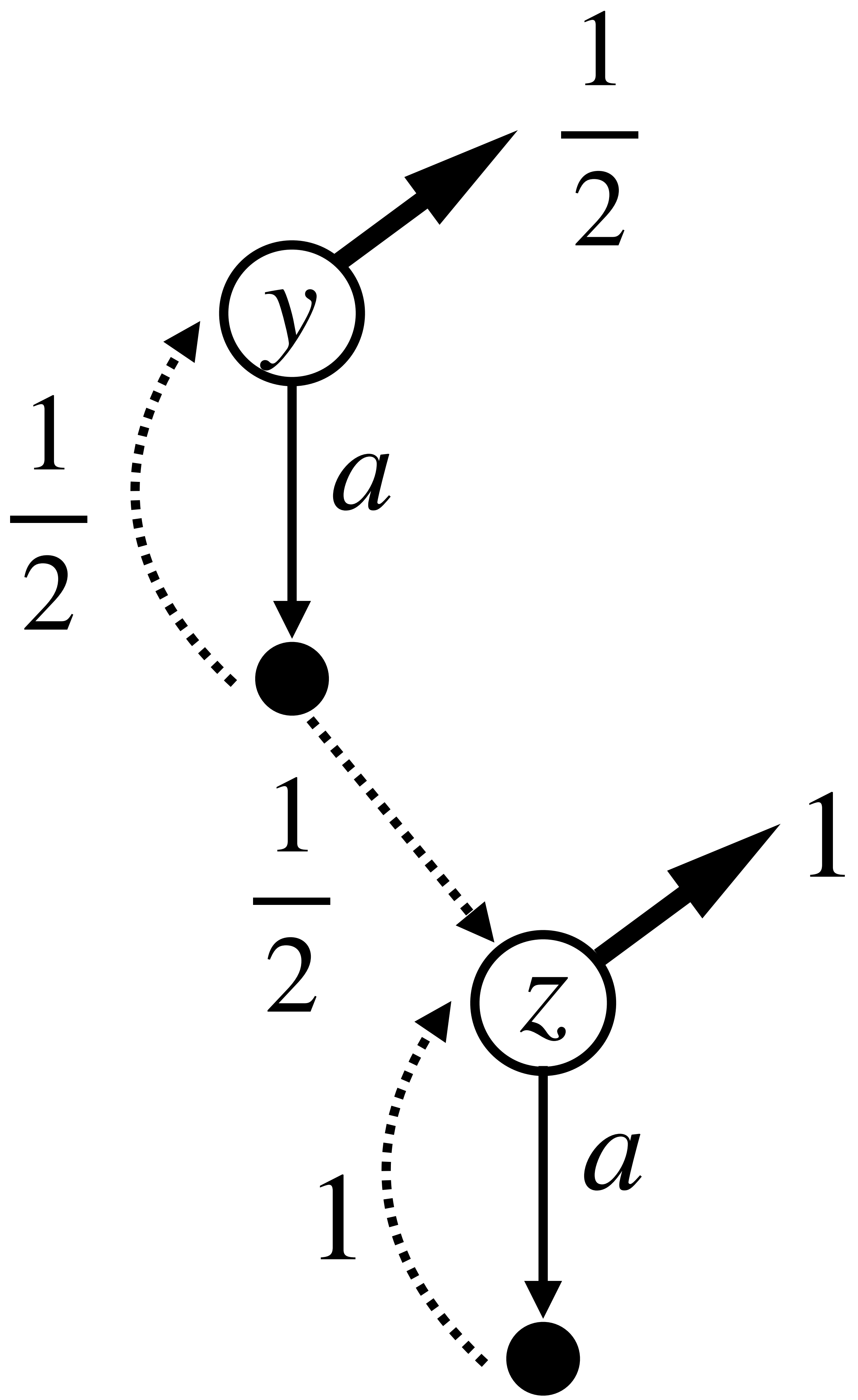
Rabin probabilistic automata





$$(X, X \to [0,1] \times \mathscr{D}(X)^A)$$

Rabin probabilistic automata



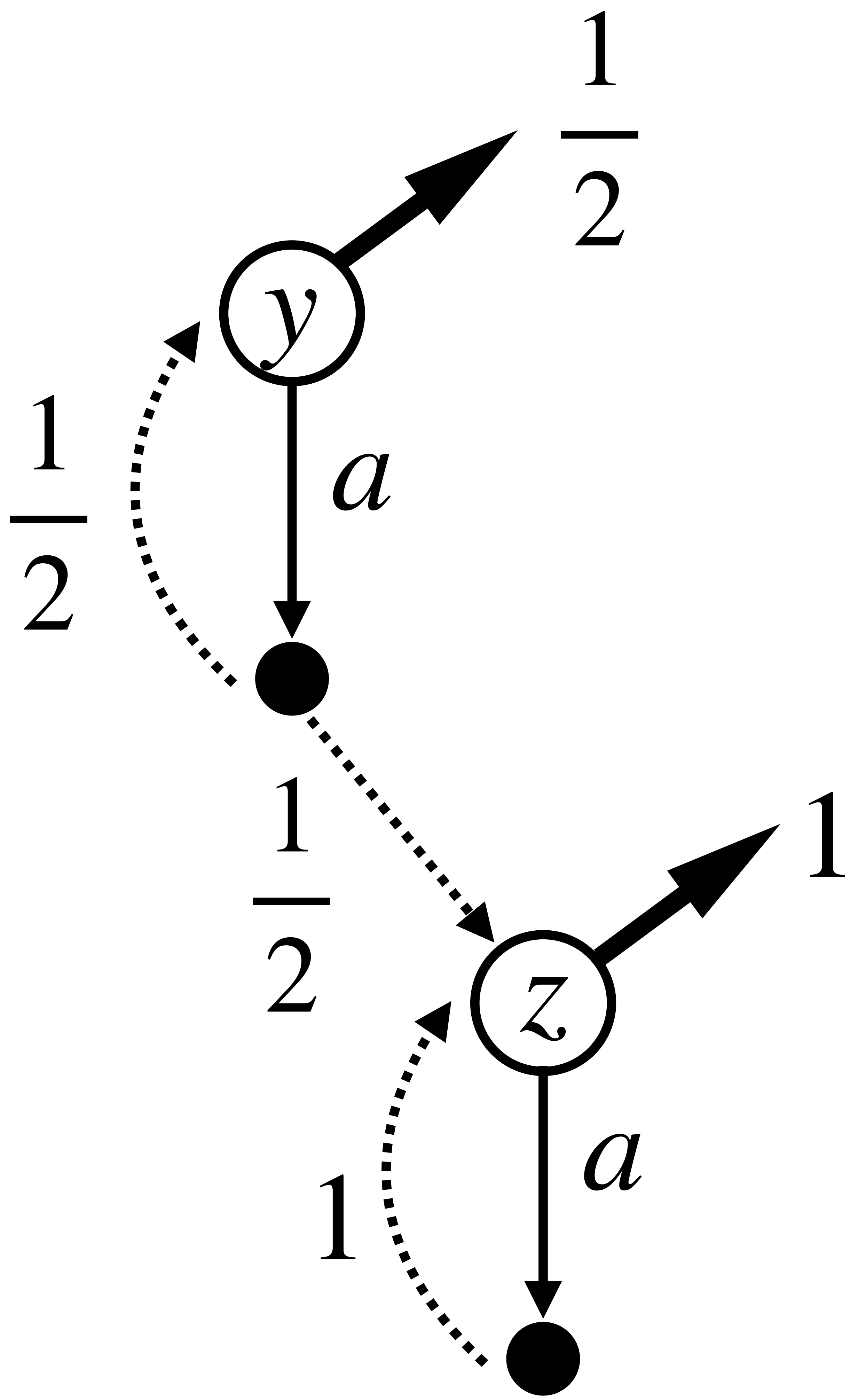
Set of states

Acceptance weight

Given a letter a, we get a distribution of successor states

$$(X, X \to [0,1] \times \mathscr{D}(X)^A)$$

Rabin probabilistic automata



Set of states

Acceptance weight

Given a letter a, we get a distribution of successor states

$$(X, X \to [0,1] \times \mathcal{D}(X)^A)$$

Language semantics

$$\mathcal{L} : X \to A^* \to [0,1]$$

The framework of Universal Coalgebra

$X \to 2 \times X^A$

The framework of Universal Coalgebra

$$X \to 2 \times X^A$$

$$X \to \mathscr{P}(1 + A \times X)$$

The framework of Universal Coalgebra

$$X \to 2 \times X^A$$

$$X \to \mathscr{P}(1 + A \times X)$$

$$X \to [0,1] \times \mathscr{D}(X)^A$$

The framework of Universal Coalgebra

$$\left.\begin{array}{l} X \to 2 \times X^A \\ X \to \mathscr{P}(1 + A \times X) \\ X \to [0,1] \times \mathscr{D}(X)^A \end{array}\right\} \quad X \to FX$$

The framework of Universal Coalgebra

$$X \to 2 \times X^A$$

$$X \to \mathscr{P}(1 + A \times X)$$

$$X \to [0,1] \times \mathscr{D}(X)^A$$

$$\Bigg\} \quad X \to FX$$

Endofunctor F describes one-step dynamics of the system

The framework of Universal Coalgebra

$$X \to 2 \times X^A$$
$$X \to \mathscr{P}(1 + A \times X)$$
$$X \to [0,1] \times \mathscr{D}(X)^A$$

$$\left.\right\} \quad X \to FX$$

Endofunctor F describes one-step dynamics of the system

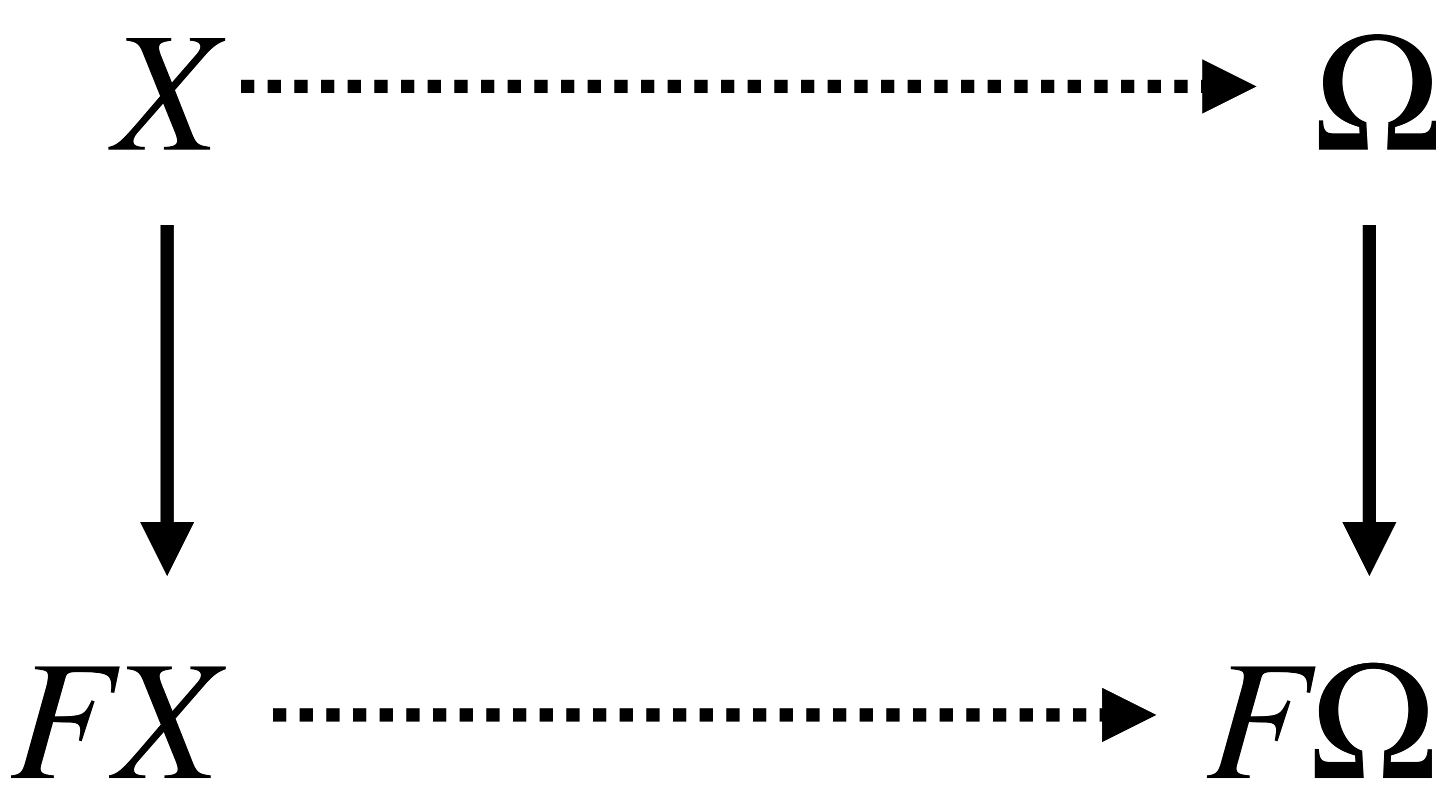
$$\begin{array}{c} X \\ \downarrow \\ FX \end{array}$$

The framework of Universal Coalgebra

$$X \to 2 \times X^A$$
$$X \to \mathscr{P}(1 + A \times X)$$
$$X \to [0,1] \times \mathscr{D}(X)^A$$

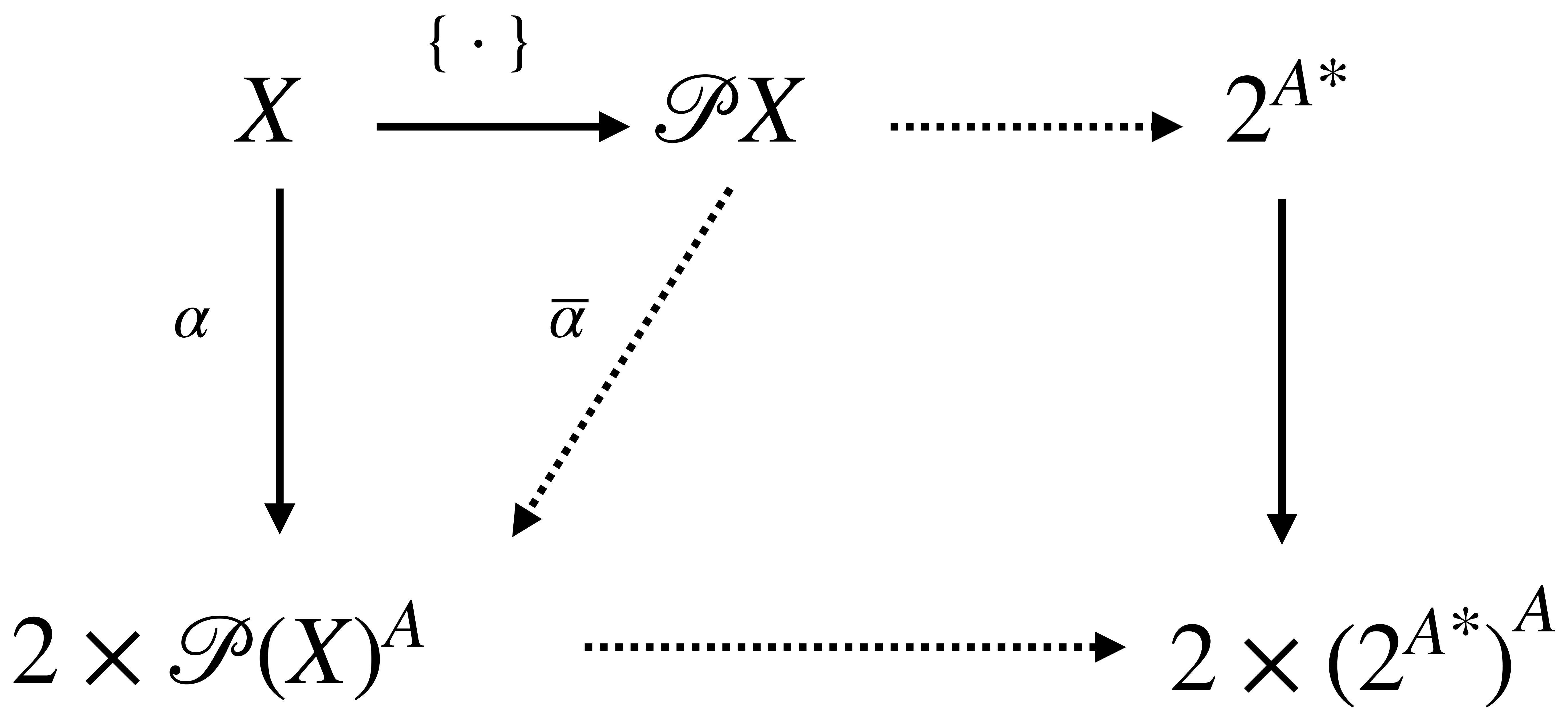
$\Big\}\; X \to FX$

Endofunctor F describes one-step dynamics of the system



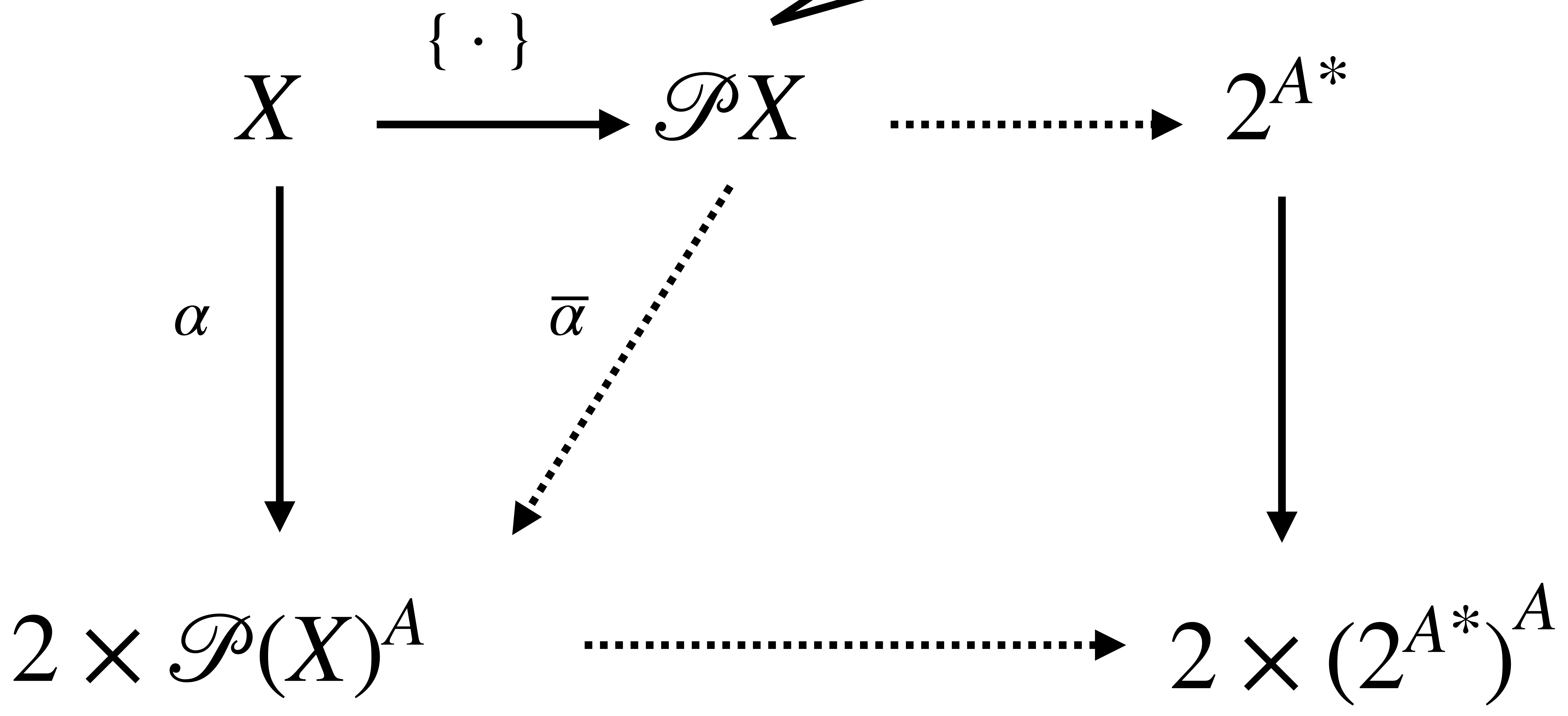
Final coalgebra: canonical domain for **branching-time semantics.** Exists under mild size constraints.

NFA determinisation

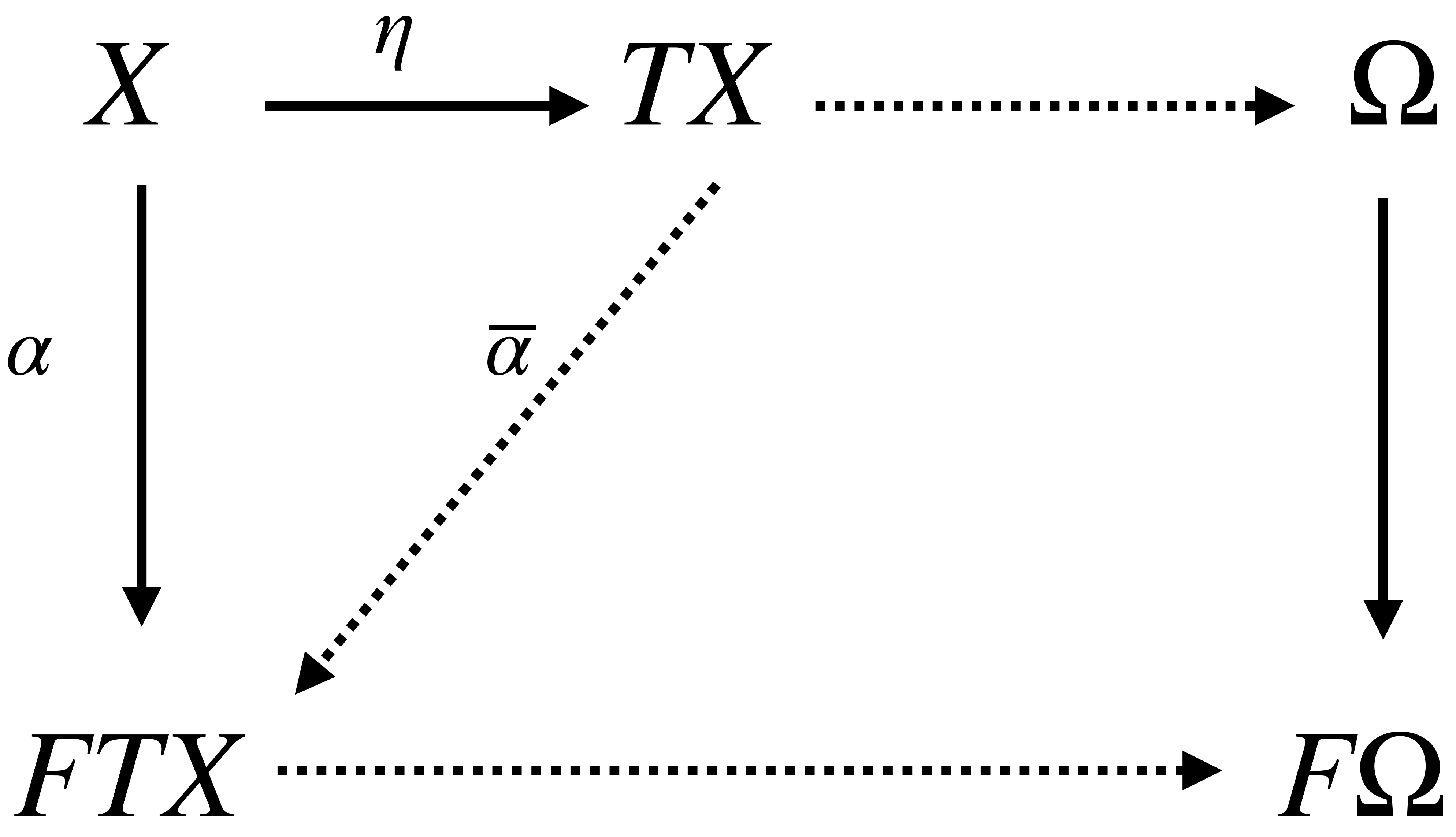


NFA determinisation

State space has extra structure of a **semilattice**

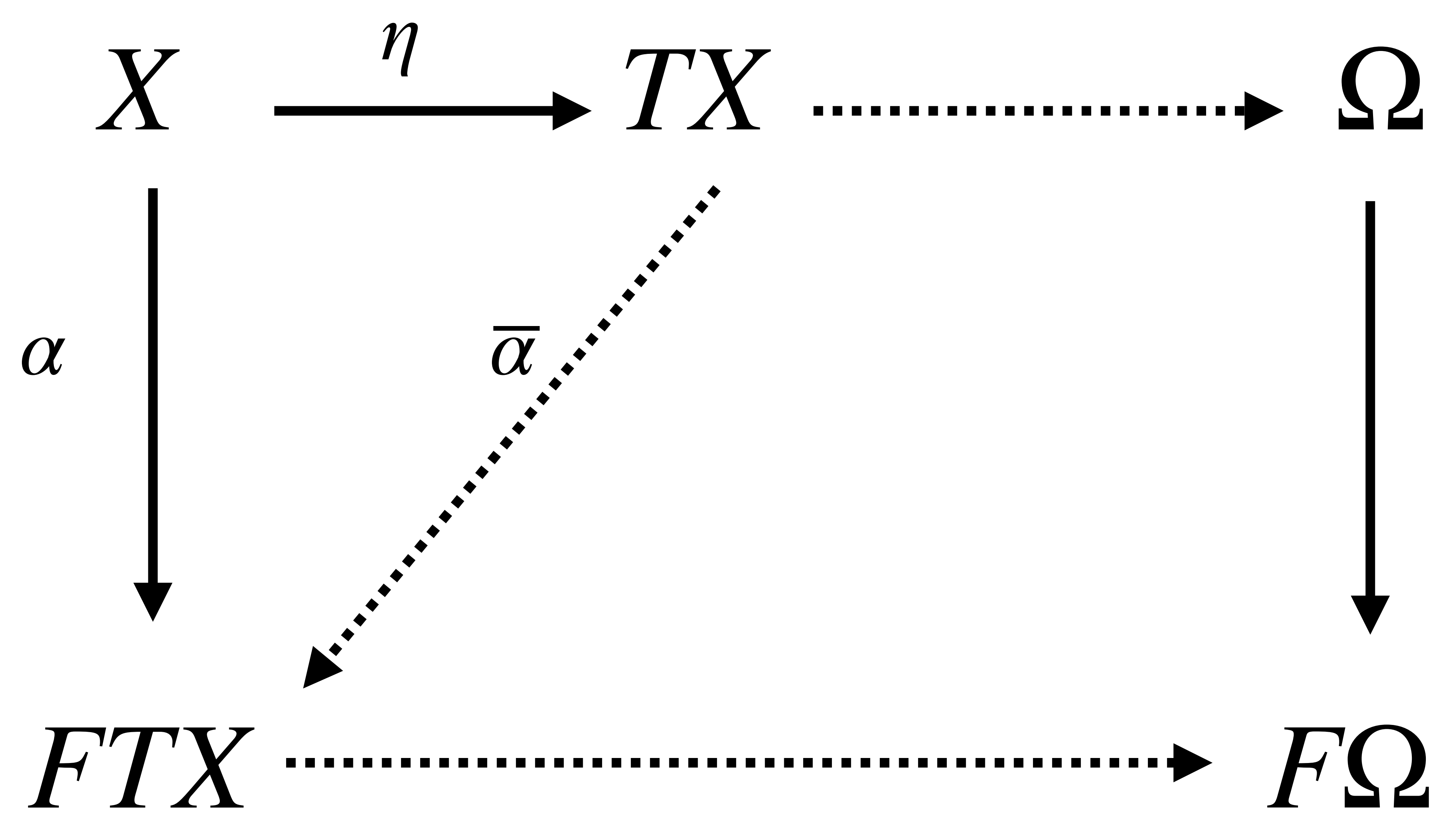


Generalized determinisation



Silva, A., Bonchi, F., Bonsangue, M. and Rutten, J. (2013). Generalizing determinization from automata to coalgebras.

Generalized determinisation

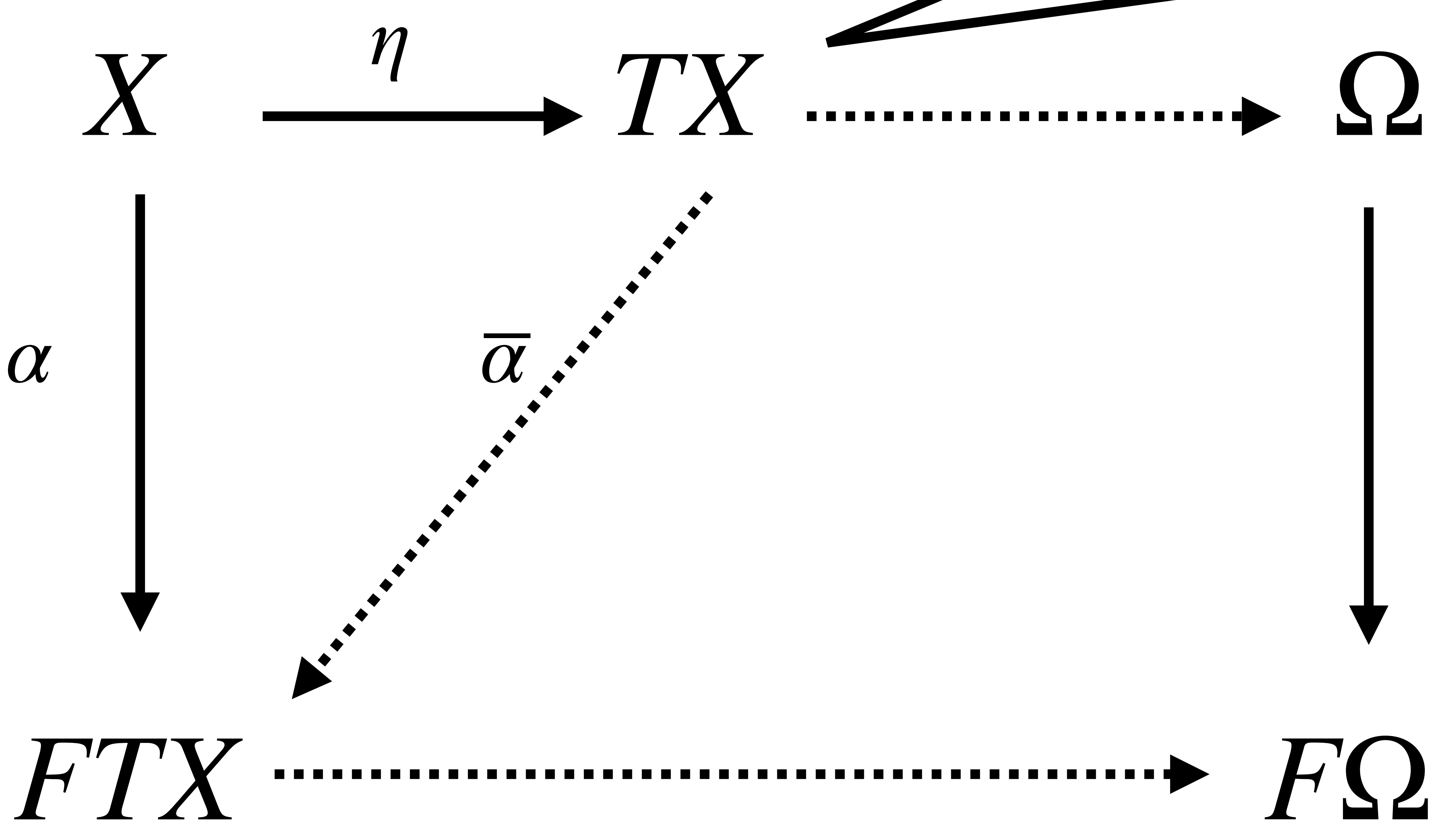


T is a monad, that interacts with F via a distributive law $\lambda : TF \Rightarrow FT$

Silva, A., Bonchi, F., Bonsangue, M. and Rutten, J. (2013). Generalizing determinization from automata to coalgebras.

Generalized determinisation

Extra structure of an algebra for a monad:
$\mu_X : TTX \to TX$

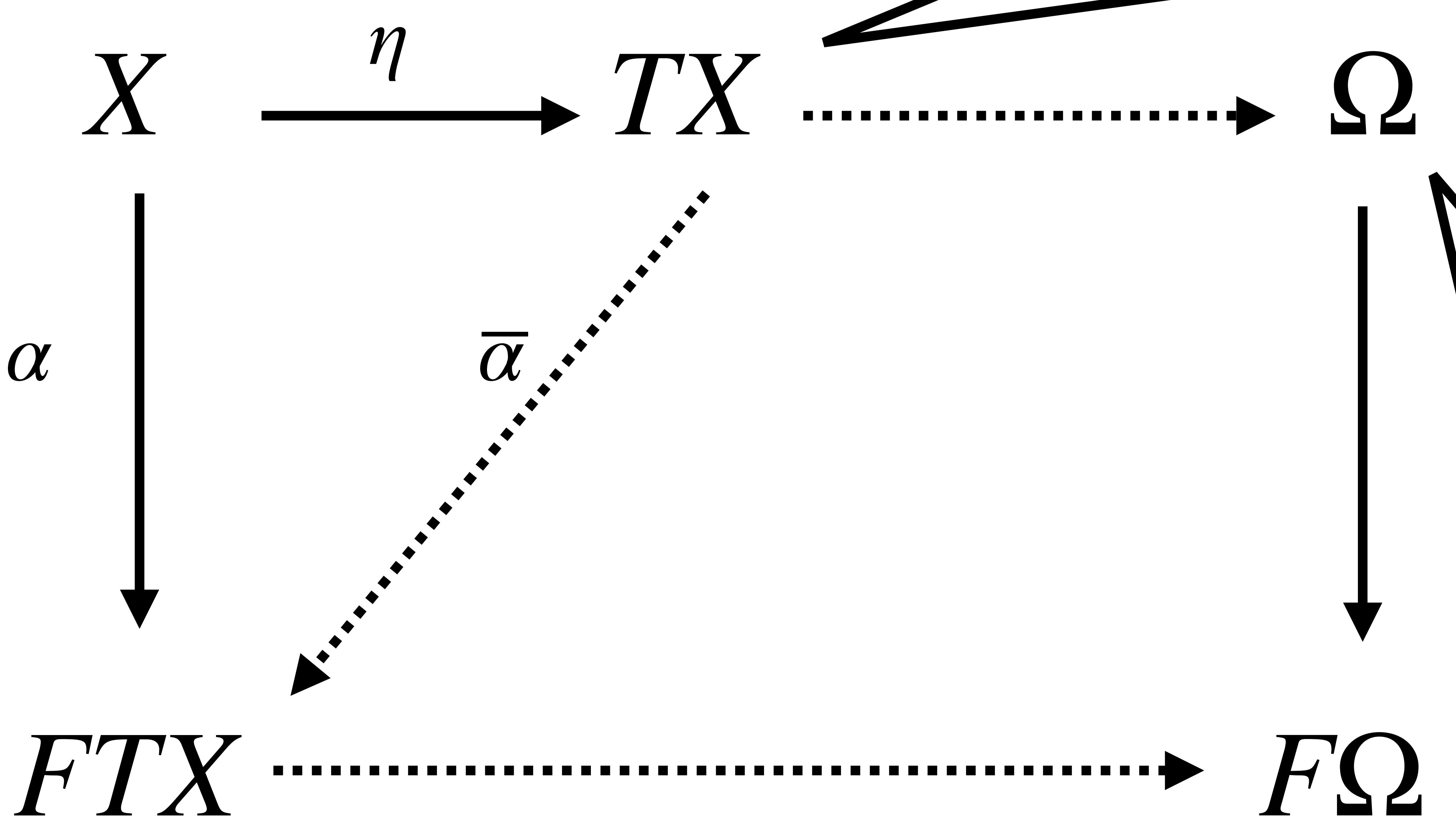


T is a monad, that interacts with F via a distributive law $\lambda : TF \Rightarrow FT$

Silva, A., Bonchi, F., Bonsangue, M. and Rutten, J. (2013). Generalizing determinization from automata to coalgebras.

Generalized determinisation

Extra structure of an algebra for a monad:
$\mu_X : TTX \to TX$

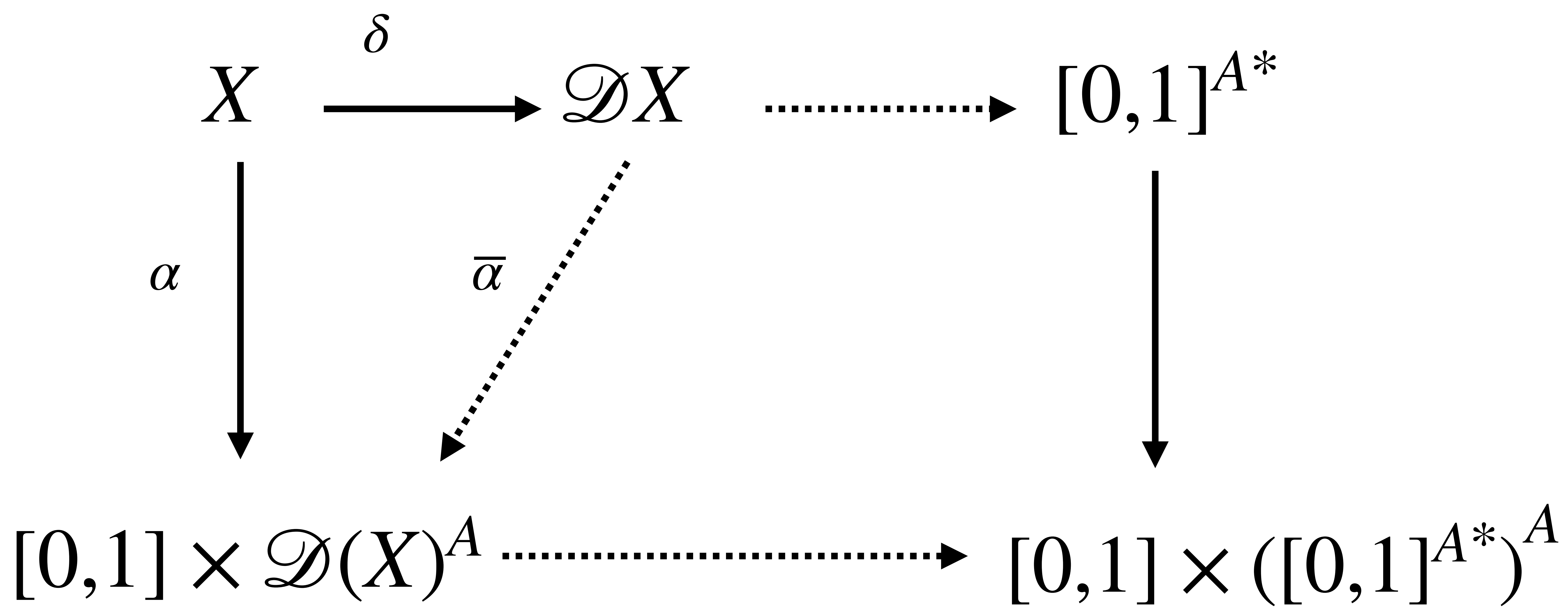


Final coalgebra for F

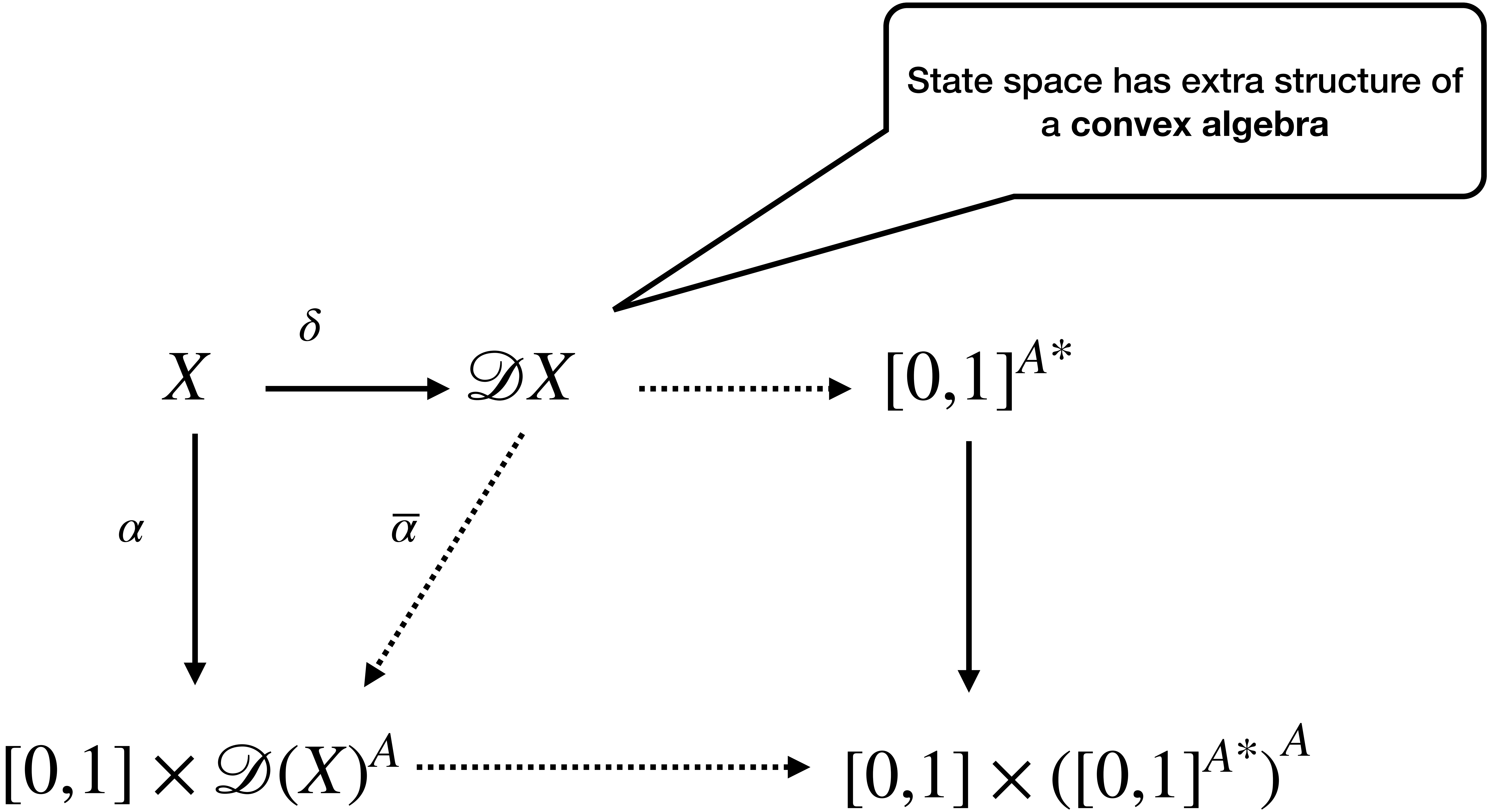
T is a monad, that interacts with F via a distributive law $\lambda : TF \Rightarrow FT$

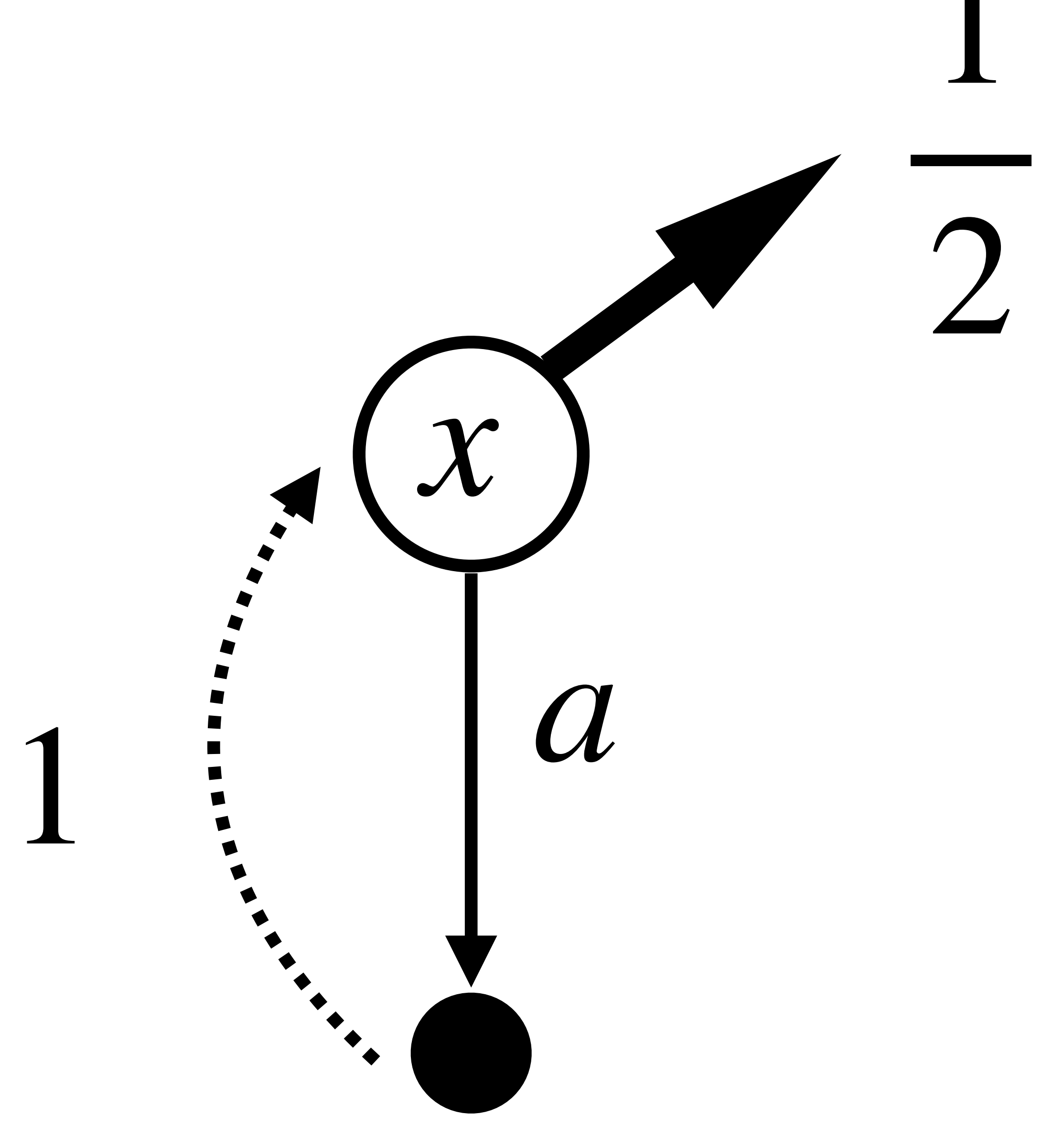
Silva, A., Bonchi, F., Bonsangue, M. and Rutten, J. (2013). Generalizing determinization from automata to coalgebras.

Back to the example of Rabin automata

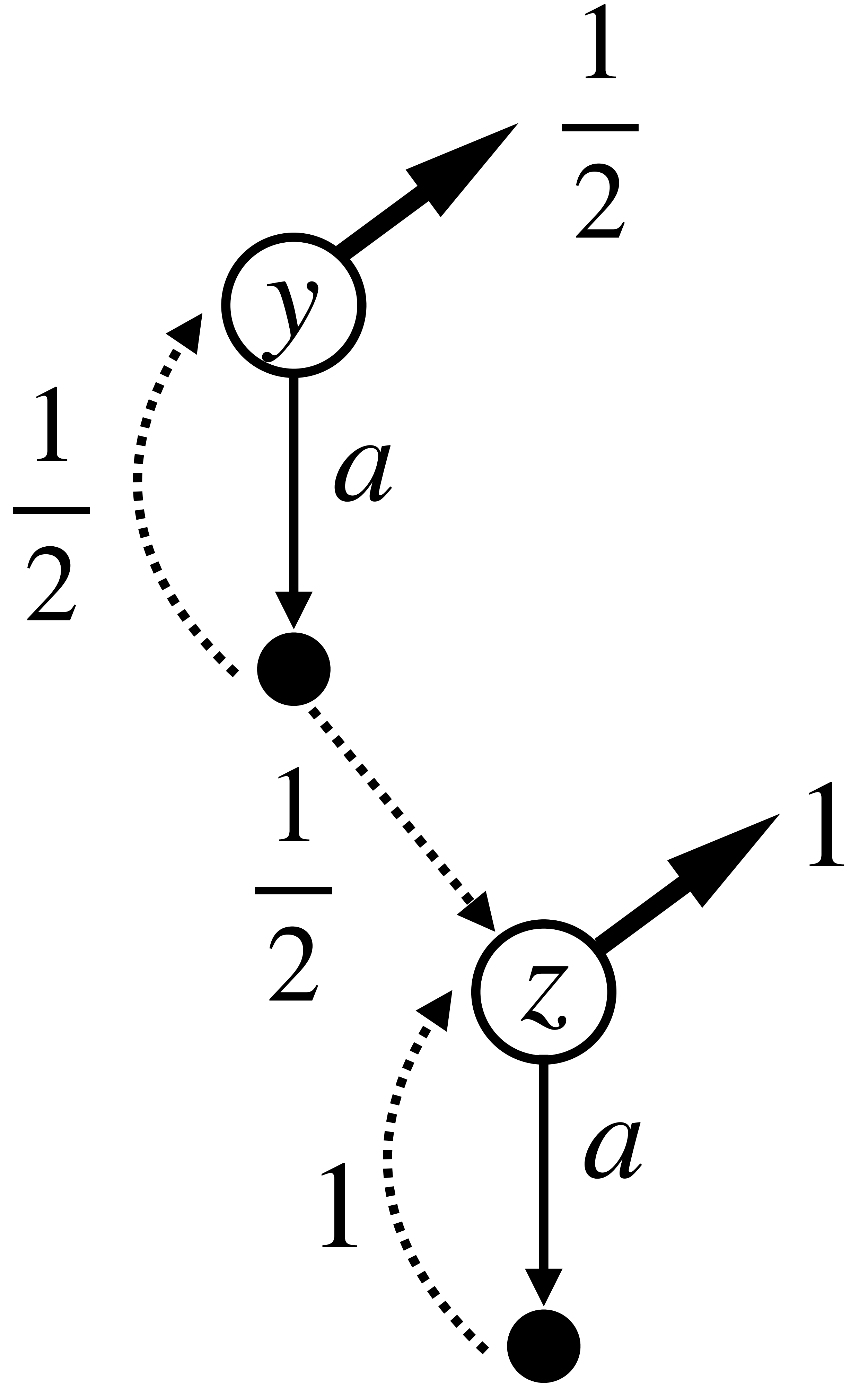


X
δ
$\mathcal{D}X$
$[0,1]^{A^*}$
α
$\overline{\alpha}$
$[0,1] \times \mathcal{D}(X)^A$
$[0,1] \times ([0,1]^{A^*})^A$

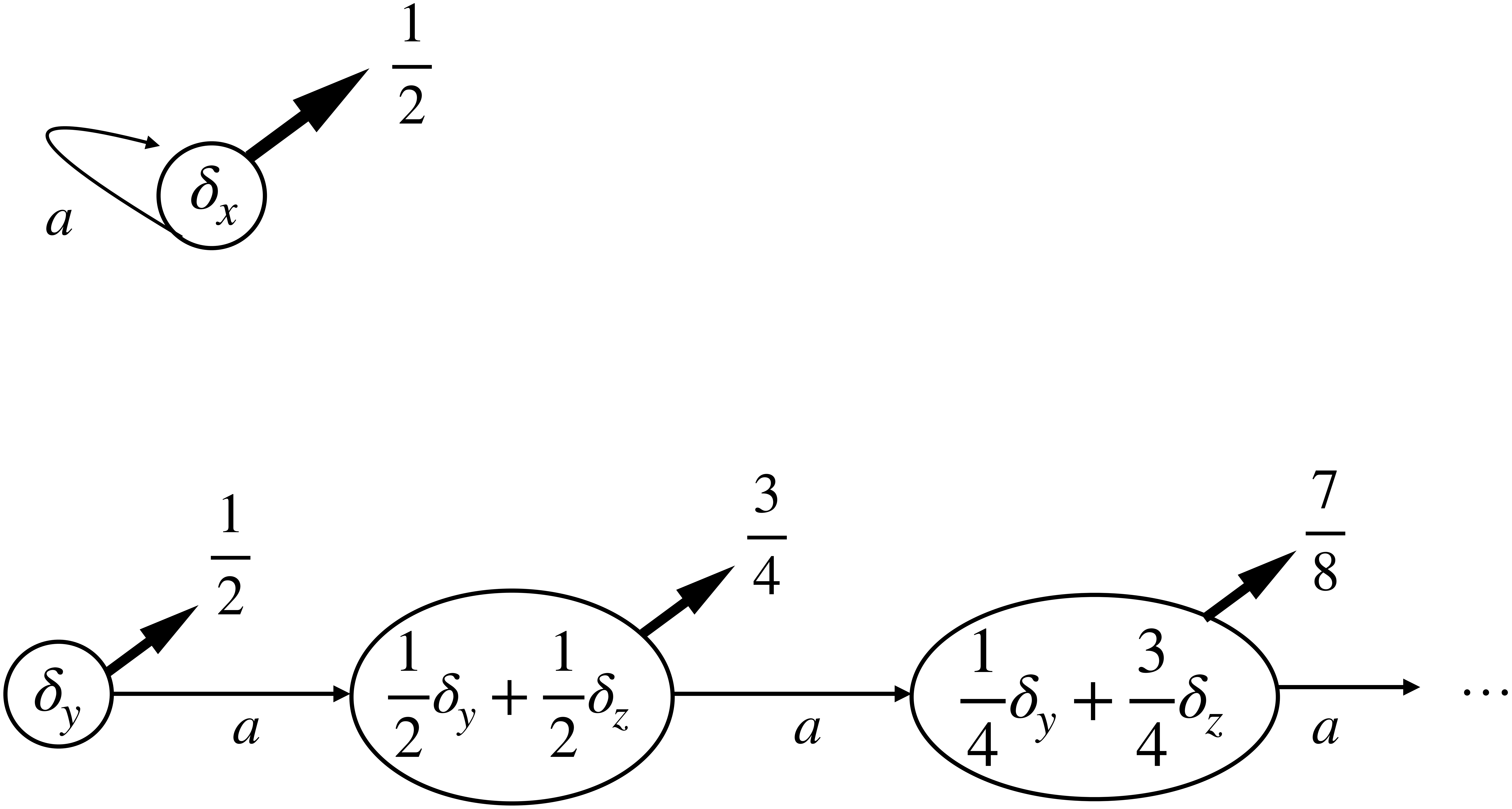

State space has extra structure of a **convex algebra**
X
δ
𝒟X
[0,1]^{A*}
α
ᾱ
[0,1] × 𝒟(X)^A
[0,1] × ([0,1]^{A*})^A



1/2
x
a
1

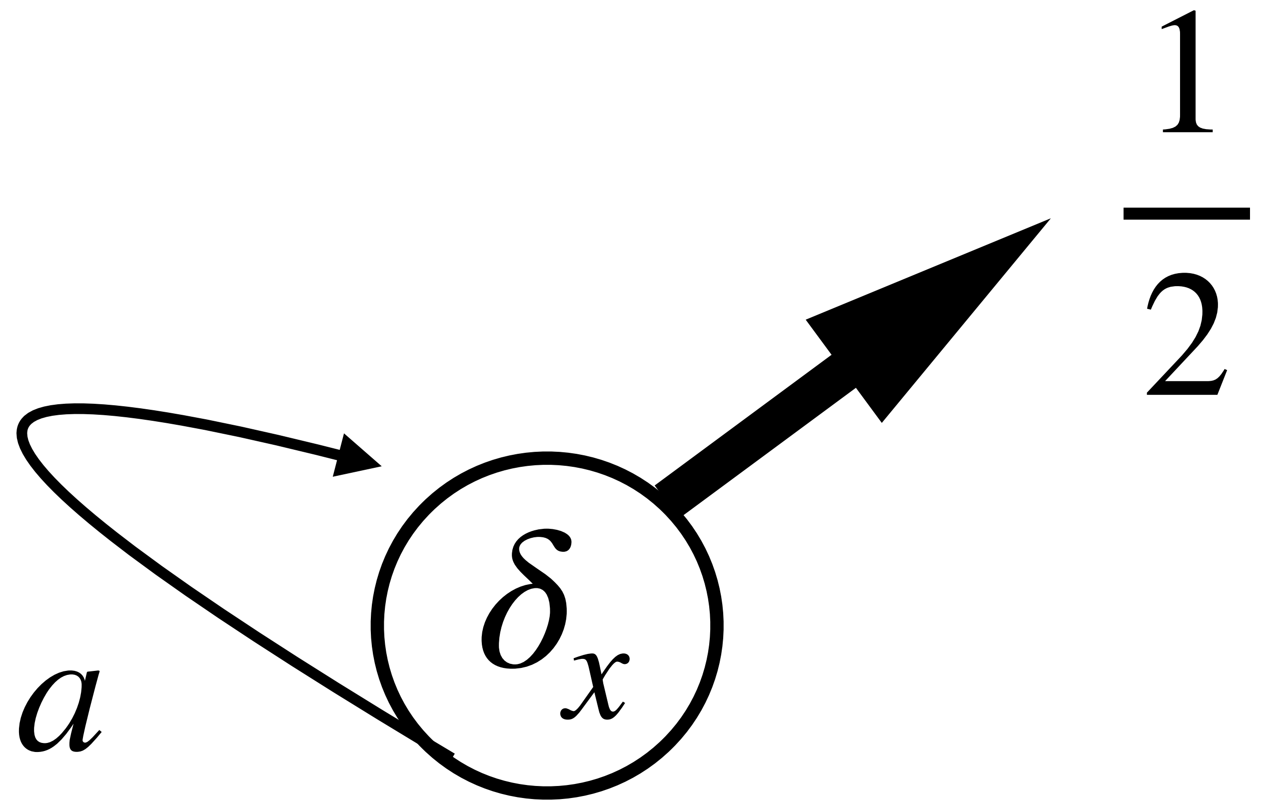


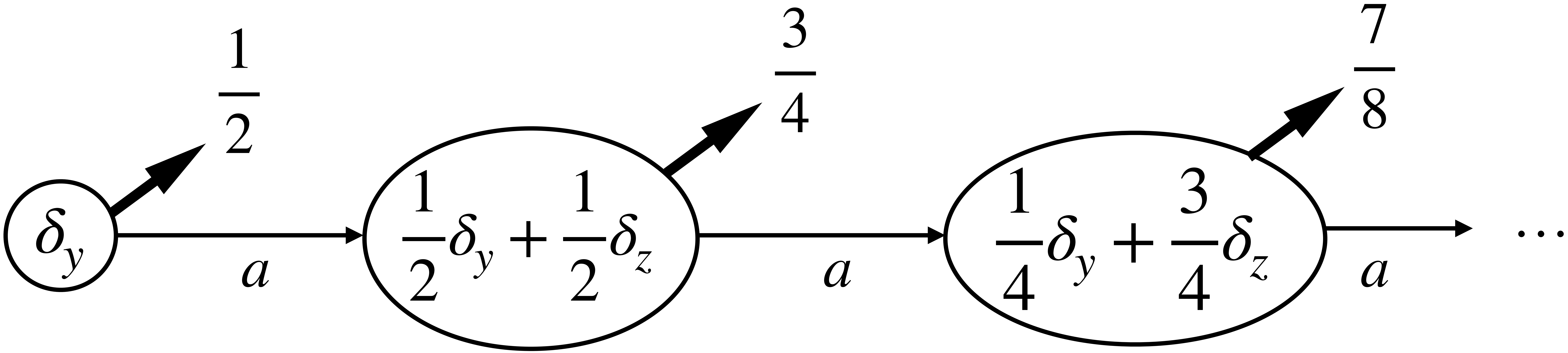
1/2
y
1/2
a
1/2
1
z
a
1

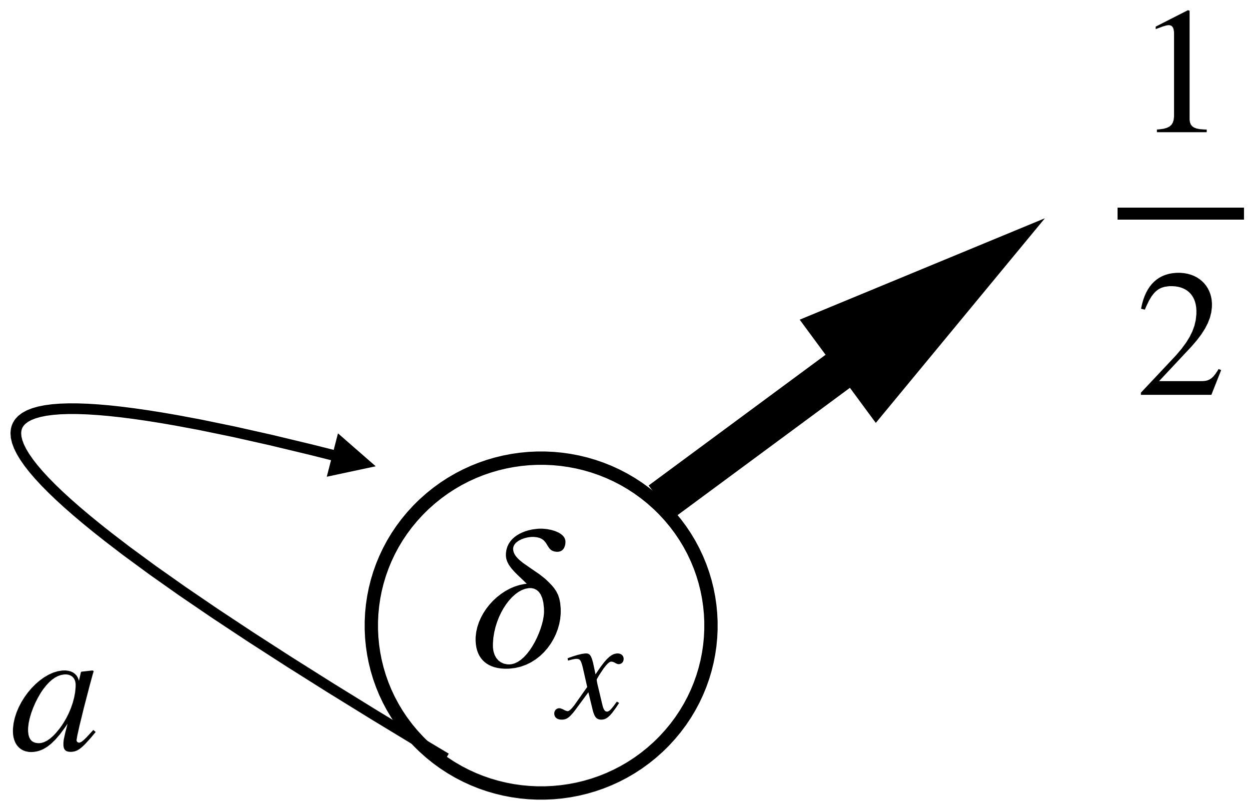


1/2
a
δ_x
1/2
3/4
7/8
δ_y
1/2 δ_y + 1/2 δ_z
1/4 δ_y + 3/4 δ_z
a
a
a
...

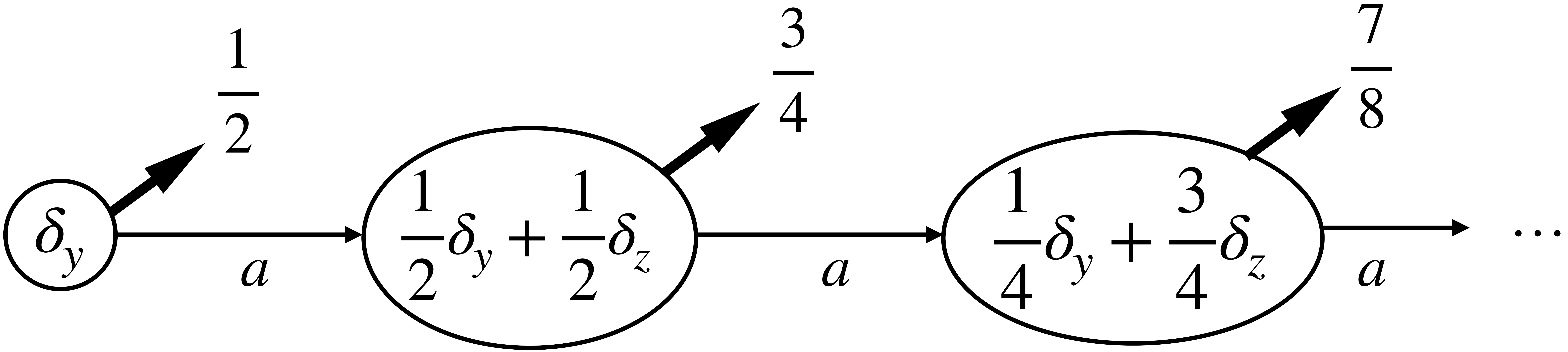
Each state assigns a weight (expected payoff) to a word







	ϵ	a	aa	$\cdots$
δ_x	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\cdots$
δ_y	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\cdots$



Behavioural distances via lifting

$F\colon \mathsf{Set} \to \mathsf{Set}$

Endofunctor describing one-step behaviour

$$F \colon \mathsf{Set} \to \mathsf{Set}$$

Endofunctor describing one-step behaviour

d is a pseudometric

$$\overline{F}(X, d \colon X \times X \to [0,1]) = (FX, d^F \colon FX \times FX \to [0,1])$$

Given a pseudometric d on the set of states, we can make a **new** one: beh(d)

$$X \times X$$

Paolo Baldan, Filippo Bonchi, Henning Kerstan and König, B. (2017). Coalgebraic Behavioral Metrics.

Given a pseudometric d on the set of states, we can make a **new** one: beh(d)

$$X \times X \xrightarrow{\alpha \times \alpha} FX \times FX$$

Paolo Baldan, Filippo Bonchi, Henning Kerstan and König, B. (2017). Coalgebraic Behavioral Metrics.

Given a pseudometric d on the set of states, we can make a **new** one: beh(d)

$$X \times X \xrightarrow{\alpha \times \alpha} FX \times FX \xrightarrow{d^F} [0,1]$$

Paolo Baldan, Filippo Bonchi, Henning Kerstan and König, B. (2017). Coalgebraic Behavioral Metrics.

Given a pseudometric d on the set of states, we can make a **new** one: beh(d)

$$X \times X \xrightarrow{\alpha \times \alpha} FX \times FX \xrightarrow{d^F} [0,1]$$

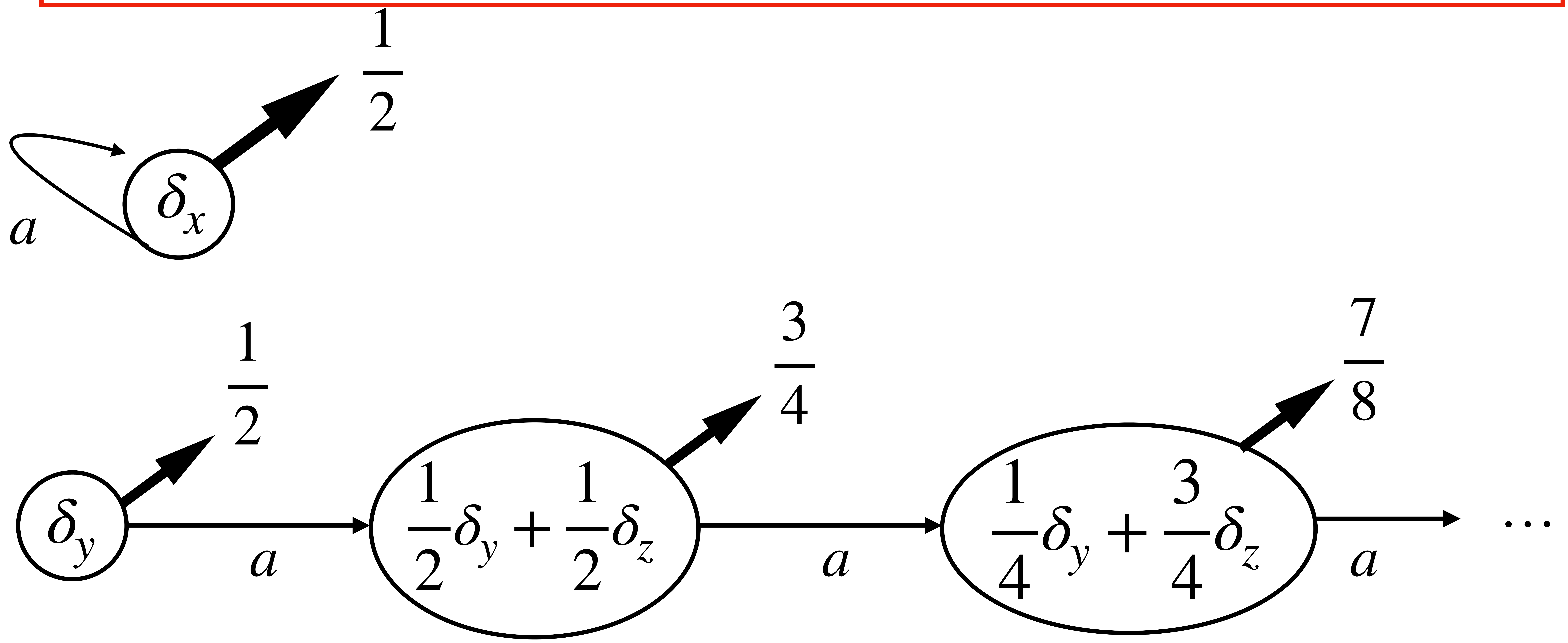
Least fixpoint μbeh gives as the behavioural distance

Paolo Baldan, Filippo Bonchi, Henning Kerstan and König, B. (2017). Coalgebraic Behavioral Metrics.

$$F = [0,1] \times X^A$$

$$d^F(\langle o_1, t_1 \rangle, \langle o_1, t_2 \rangle) = \max\{\, |o_1 - o_2|\,, \max_{a \in A} d(t_1(a), t_2(a))\}$$

Is it true that $d(\delta_x, \delta_y) \leq \frac{1}{2}$ and $d(\delta_x, \delta_z) \leq \frac{1}{2}$?

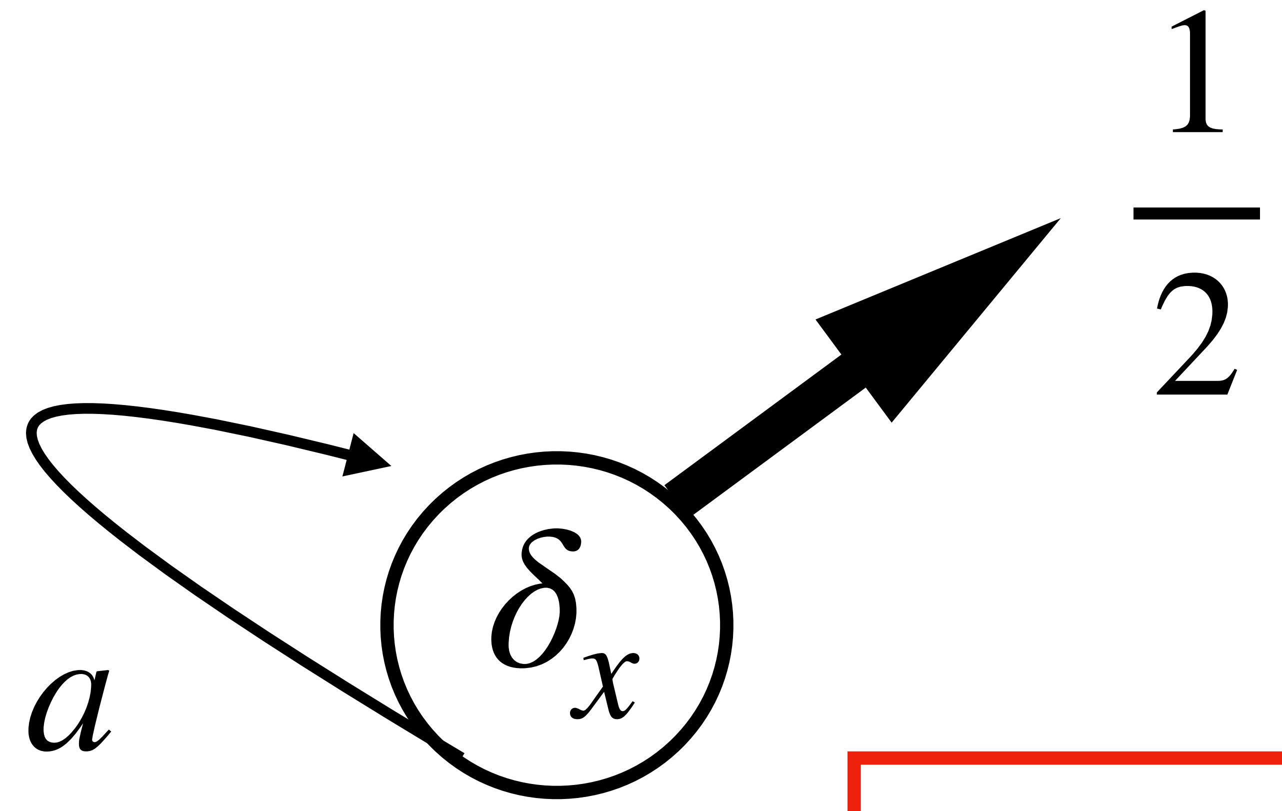


Induction proof principle

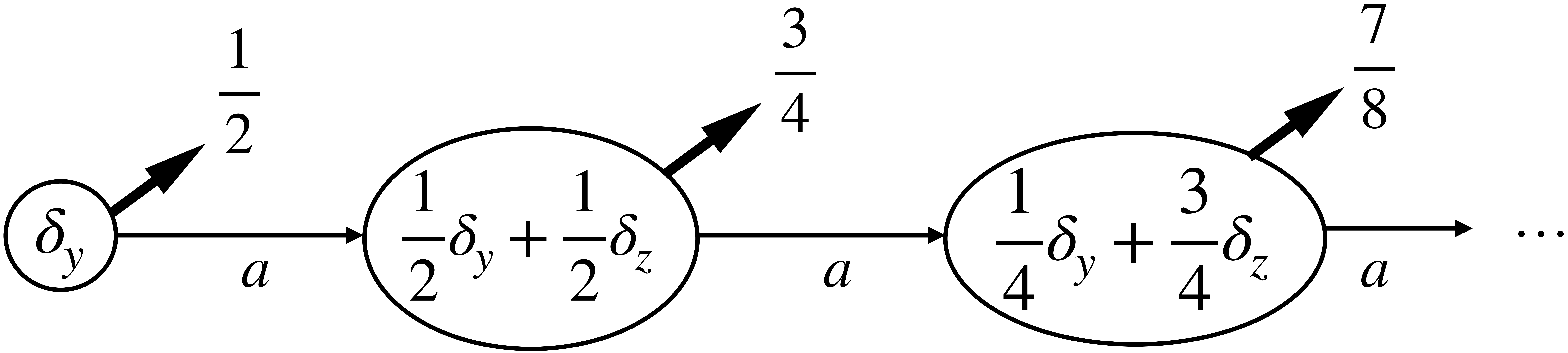
$$\frac{\mathsf{beh}(d) \leq d}{\mu\mathsf{beh} \leq d}$$

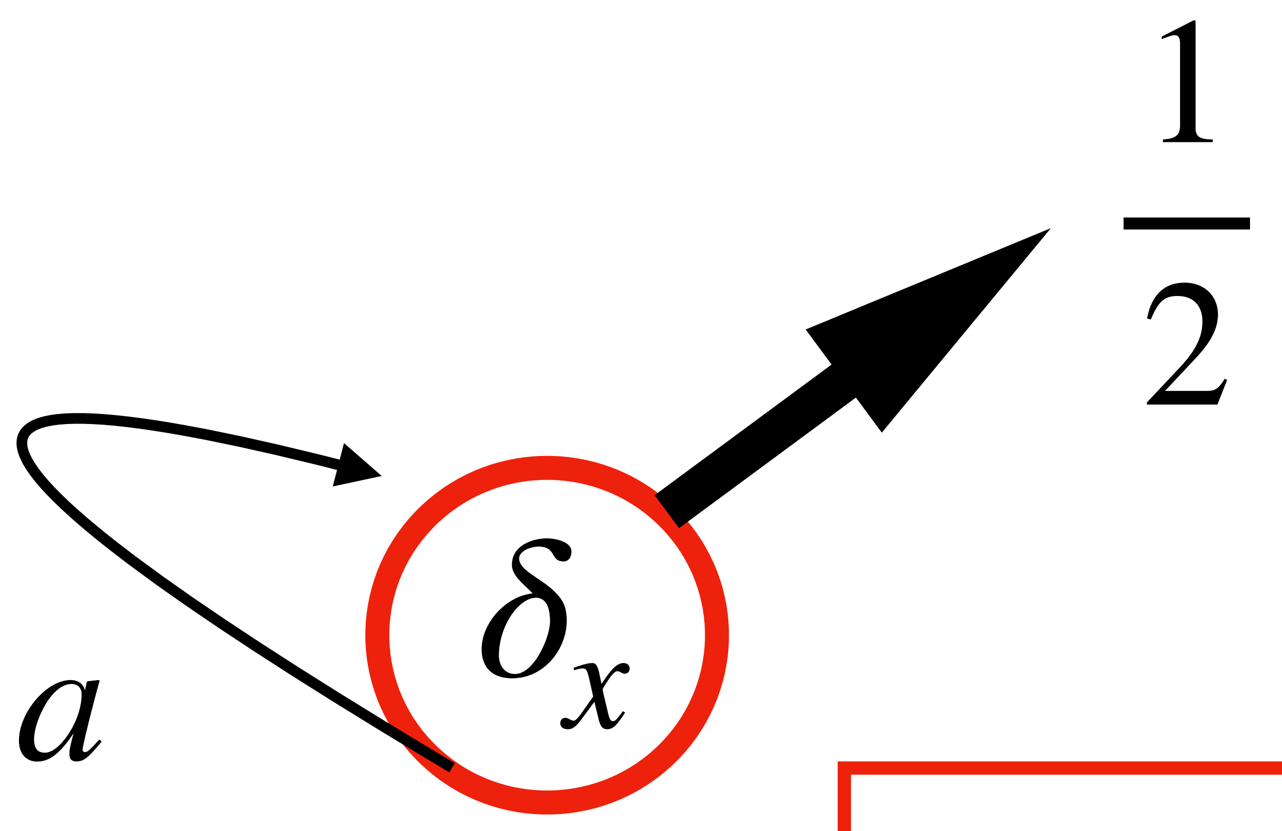
Witness of the upper bound

$$d(\delta_x, \delta_y) = \frac{1}{2}, d(\delta_x, \delta_z) = \frac{1}{2}, d(p, q) = 1$$

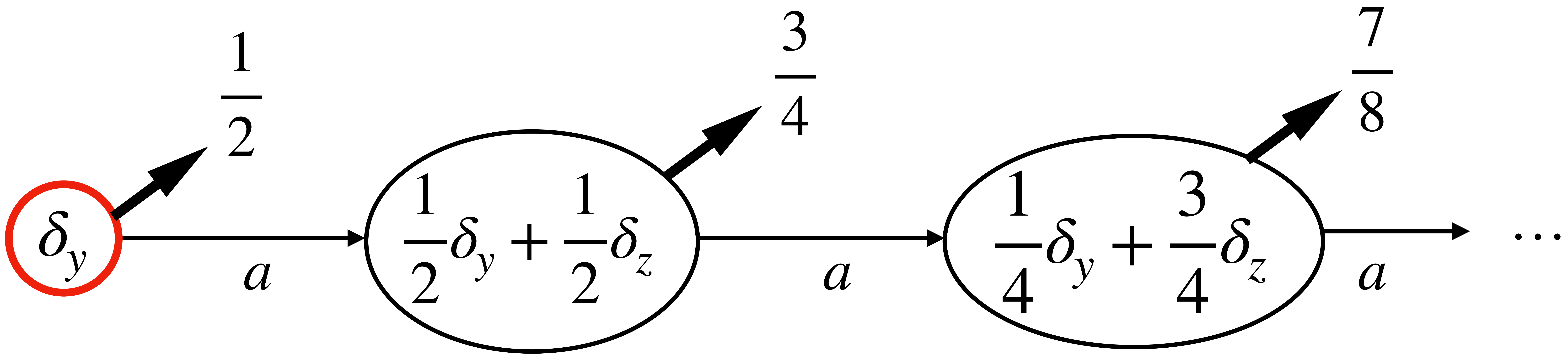


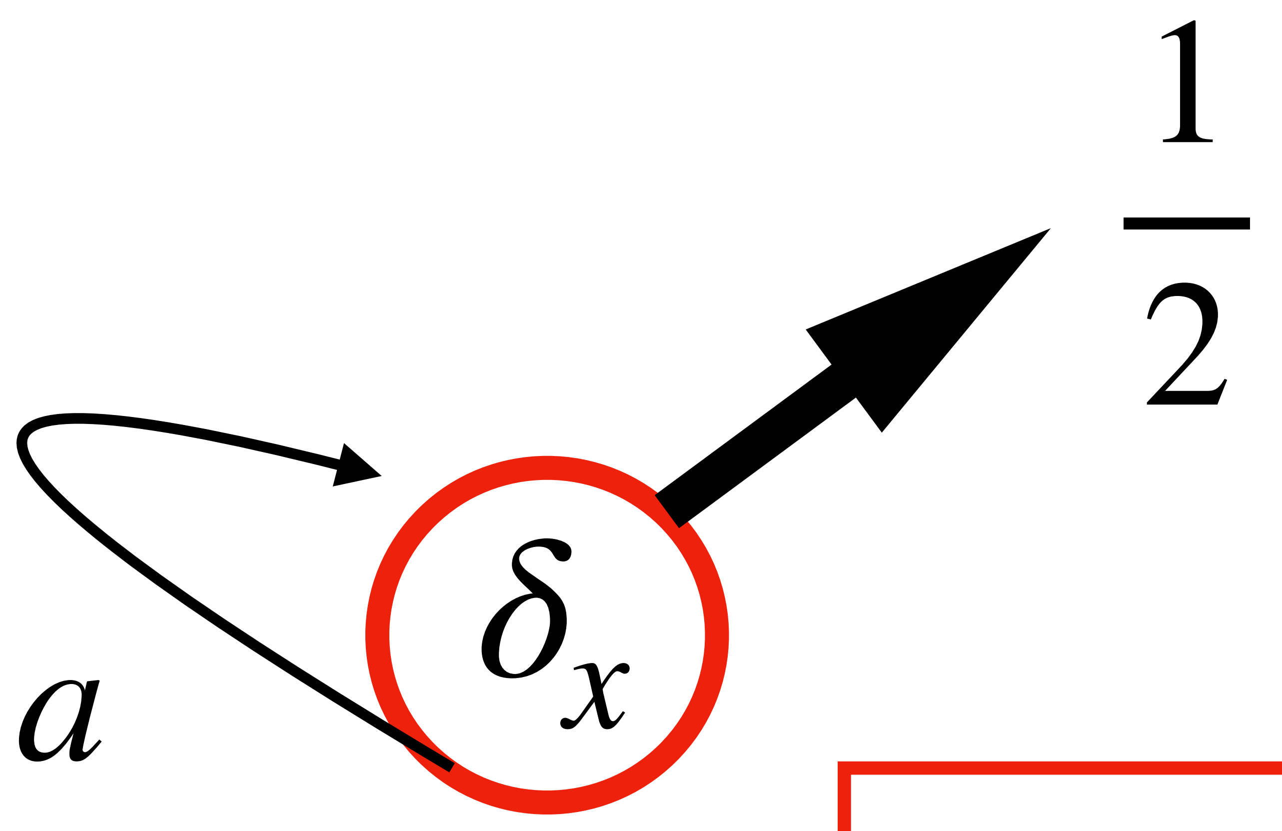
$$\mathsf{beh}(d)(\delta_x, \delta_y) = \max\{\frac{1}{2} - \frac{1}{2}, d(\delta_x, \frac{1}{2}\delta_y + \frac{1}{2}\delta_z)\}$$



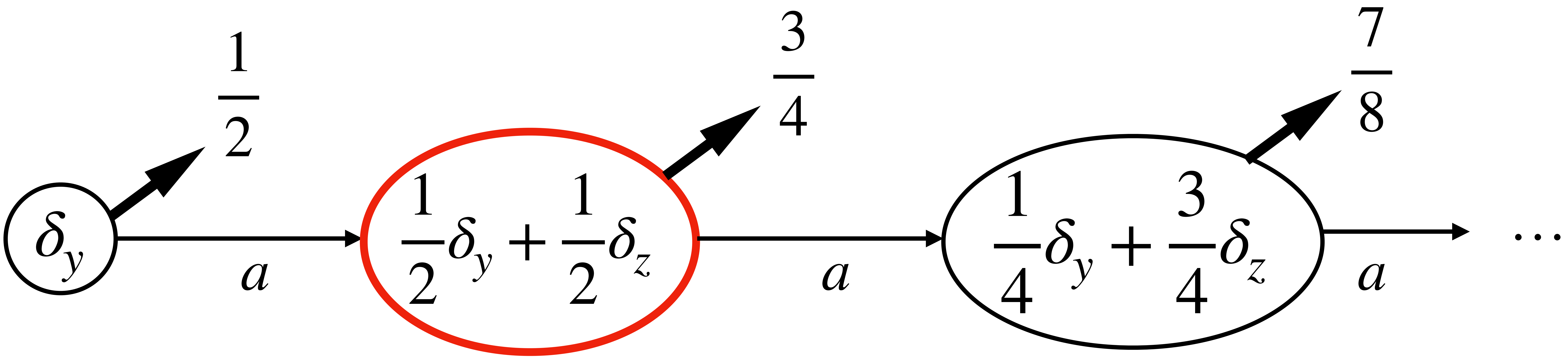


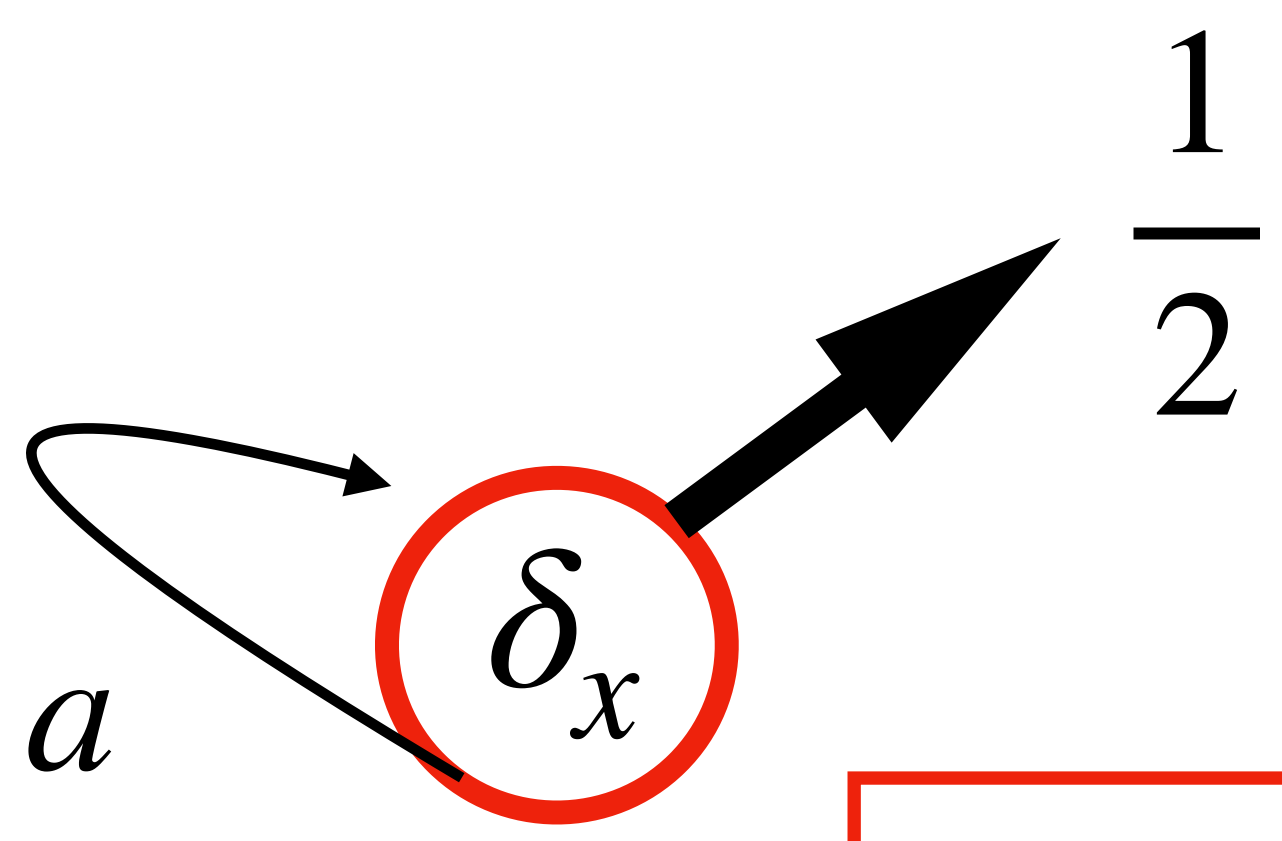
$$\mathsf{beh}(d)(\delta_x, \delta_y) = \max\{\frac{1}{2} - \frac{1}{2}, d(\delta_x, \frac{1}{2}\delta_y + \frac{1}{2}\delta_z)\}$$





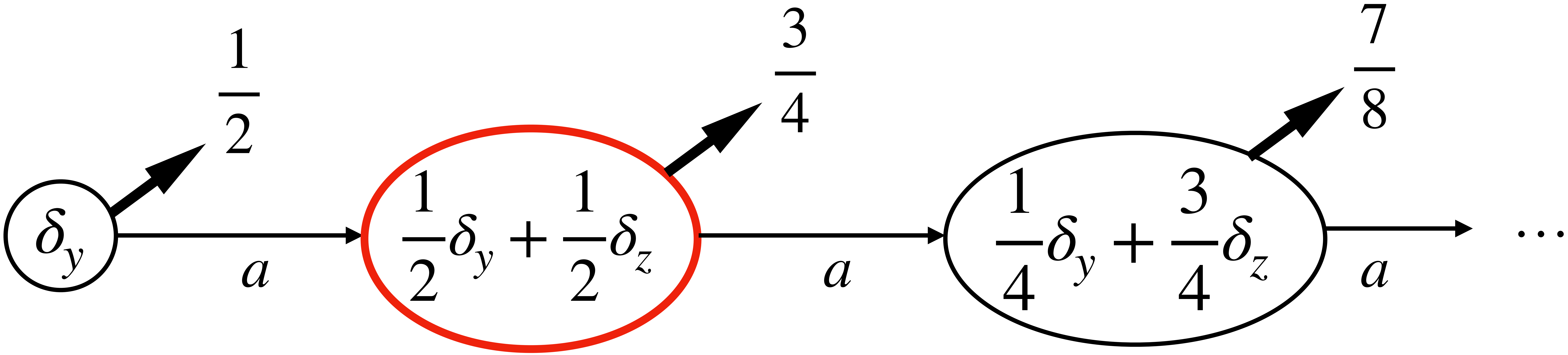
$$\mathsf{beh}(d)(\delta_x, \delta_y) = \max\{\frac{1}{2} - \frac{1}{2}, d(\delta_x, \frac{1}{2}\delta_y + \frac{1}{2}\delta_z)\}$$

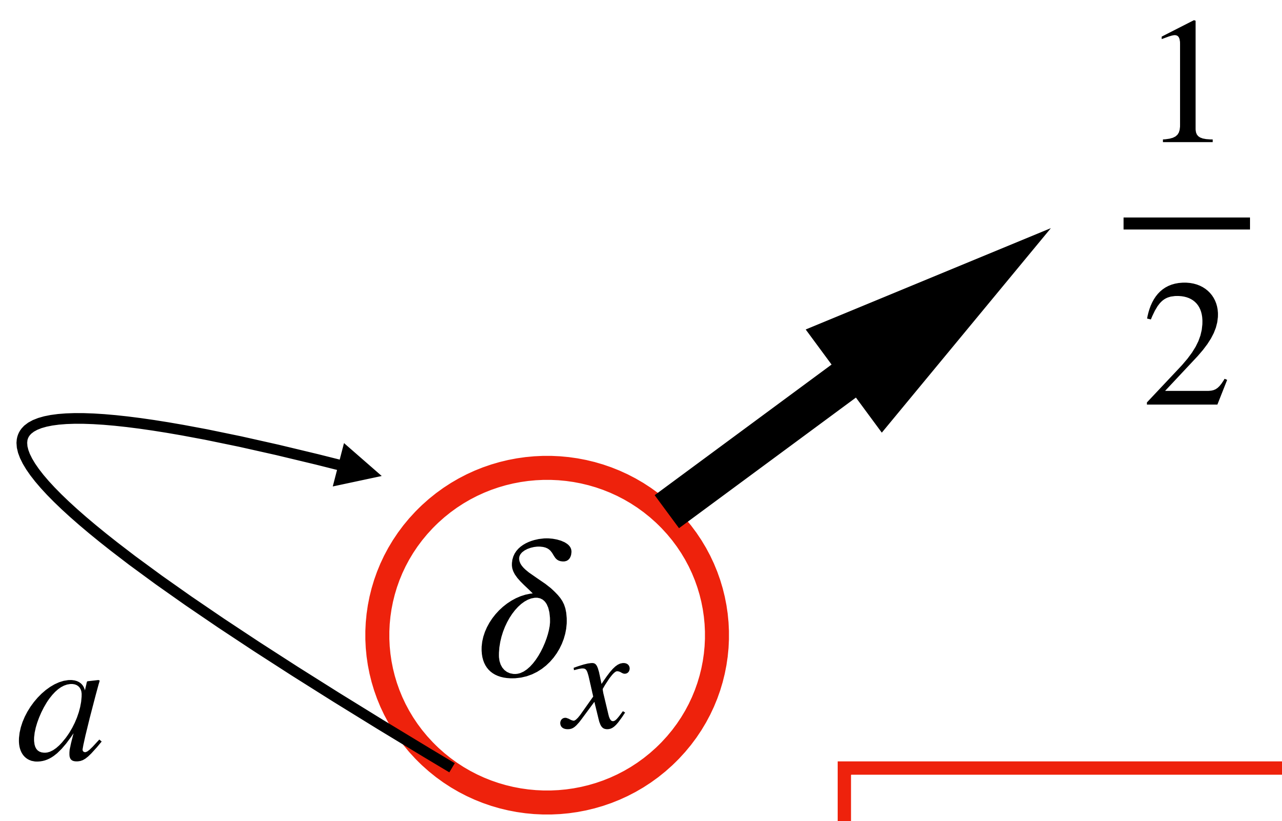




$$d(\delta_x, \delta_y) = \frac{1}{2}, d(\delta_x, \delta_z) = \frac{1}{2}, d(p, q) = 1$$

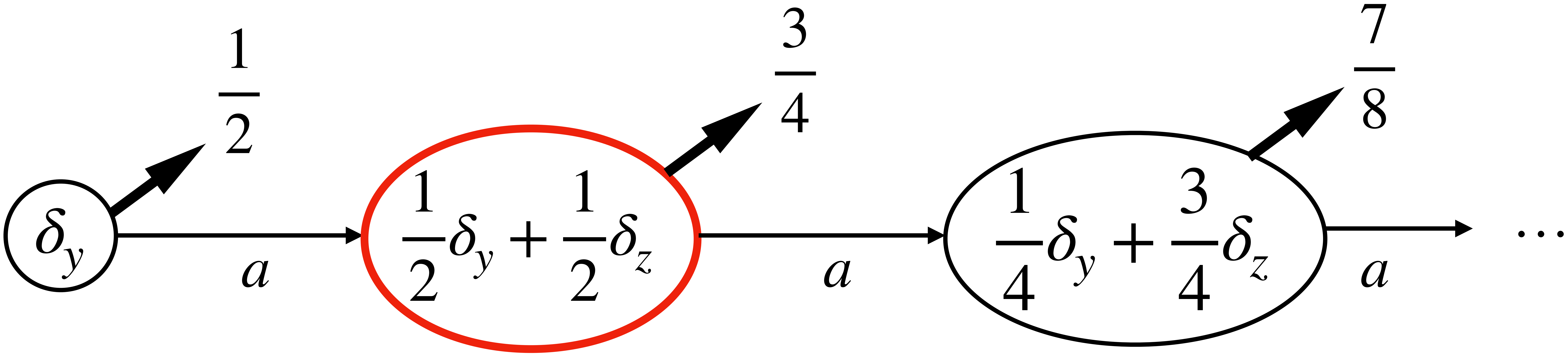
$$\mathsf{beh}(d)(\delta_x, \delta_y) = \max\{\frac{1}{2} - \frac{1}{2}, d(\delta_x, \frac{1}{2}\delta_y + \frac{1}{2}\delta_z)\}$$

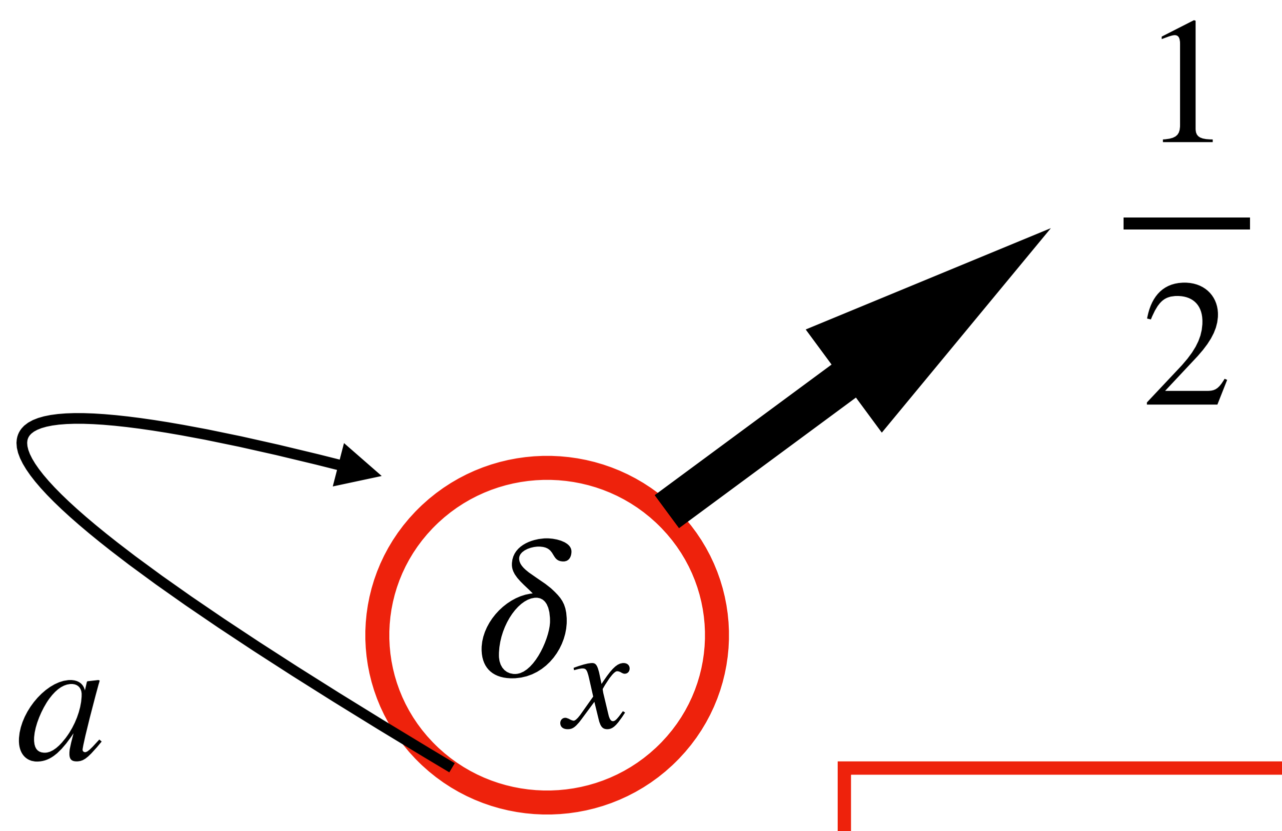




$$d(\delta_x, \delta_y) = \frac{1}{2}, d(\delta_x, \delta_z) = \frac{1}{2}, d(p, q) = 1$$

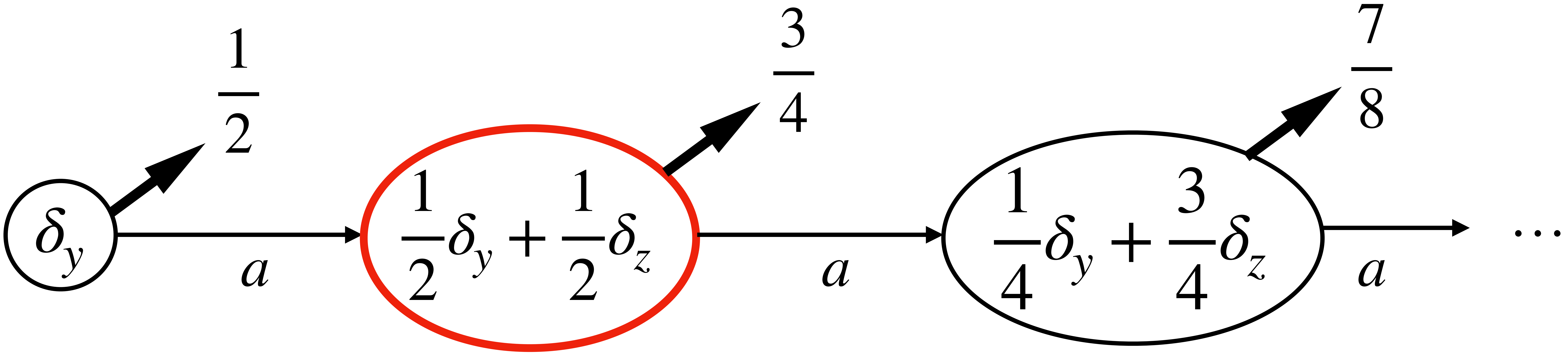
$$\mathsf{beh}(d)(\delta_x, \delta_y) = \max\{\frac{1}{2} - \frac{1}{2}, d(\delta_x, \frac{1}{2}\delta_y + \frac{1}{2}\delta_z)\}$$





$$d(\delta_x, \delta_y) = \frac{1}{2}, d(\delta_x, \delta_z) = \frac{1}{2}, d(p, q) = 1$$

$$\mathsf{beh}(d)(\delta_x, \delta_y) = \max\{\frac{1}{2} - \frac{1}{2}, d(\delta_x, \frac{1}{2}\delta_y + \frac{1}{2}\delta_z)\} = 1$$



$$d(\delta_x, \delta_y) \leq \frac{1}{2} \textbf{ and } d(\delta_x, \delta_z) \leq \frac{1}{2}$$

Chatzikokolakis, K., Gebler, D. and Palamidessi, C. (2014). Generalized bisimulation metrics.

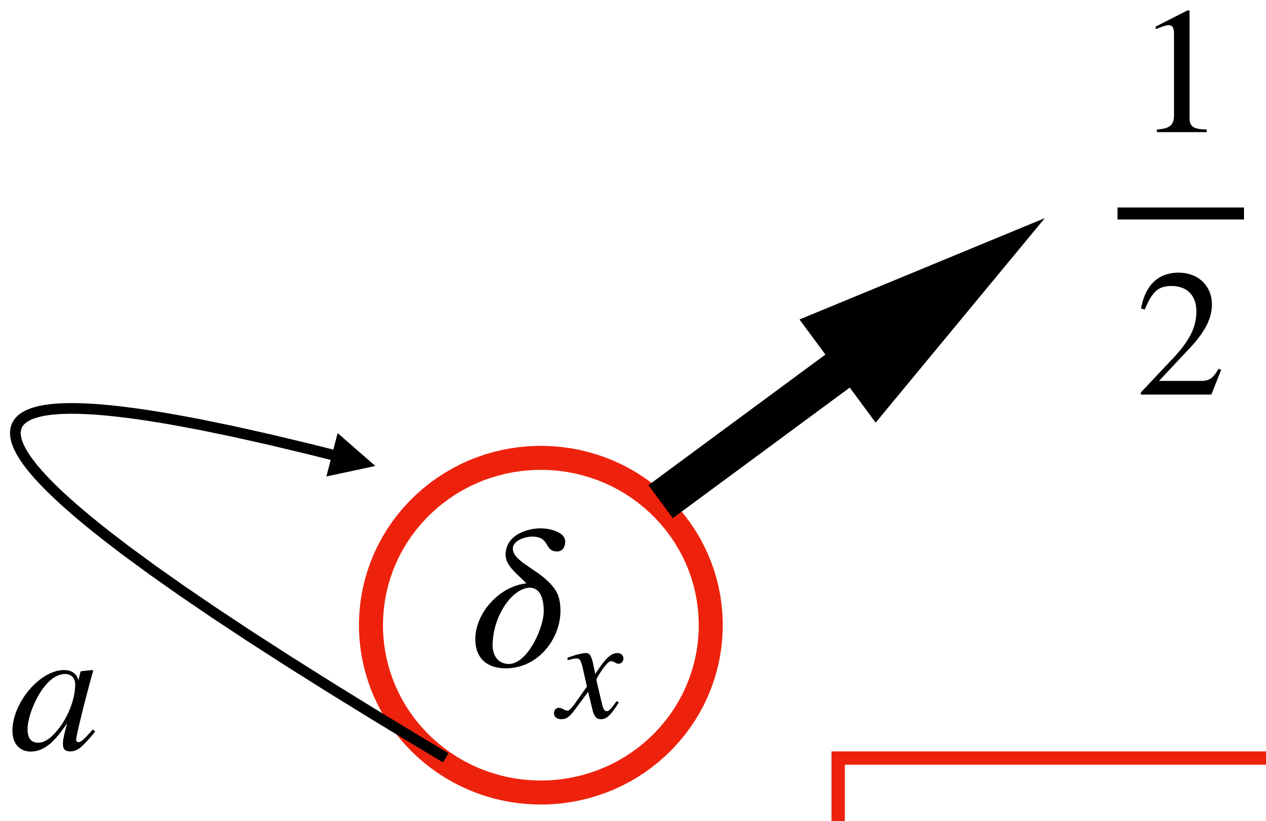
$$\frac{d(\delta_x, \delta_y) \leq \dfrac{1}{2} \textbf{ and } d(\delta_x, \delta_z) \leq \dfrac{1}{2}}{d(\delta_x, p\delta_y + (1-p)\delta_z) \leq p\dfrac{1}{2} + (1-p)\dfrac{1}{2} = \dfrac{1}{2}}$$

Chatzikokolakis, K., Gebler, D. and Palamidessi, C. (2014). Generalized bisimulation metrics.

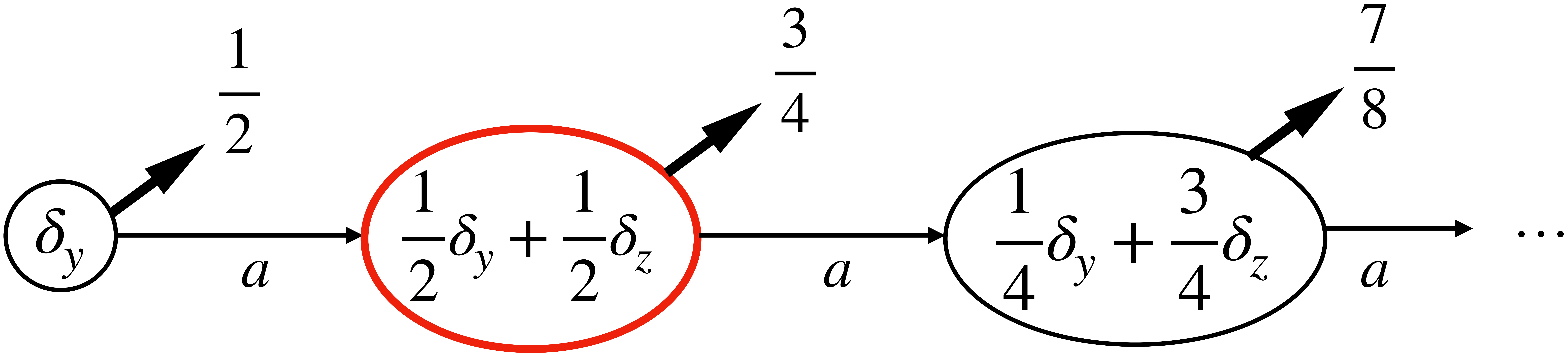
$$\frac{\mathsf{beh}(u(d)) \leq d}{\mu\mathsf{beh} \leq d}$$

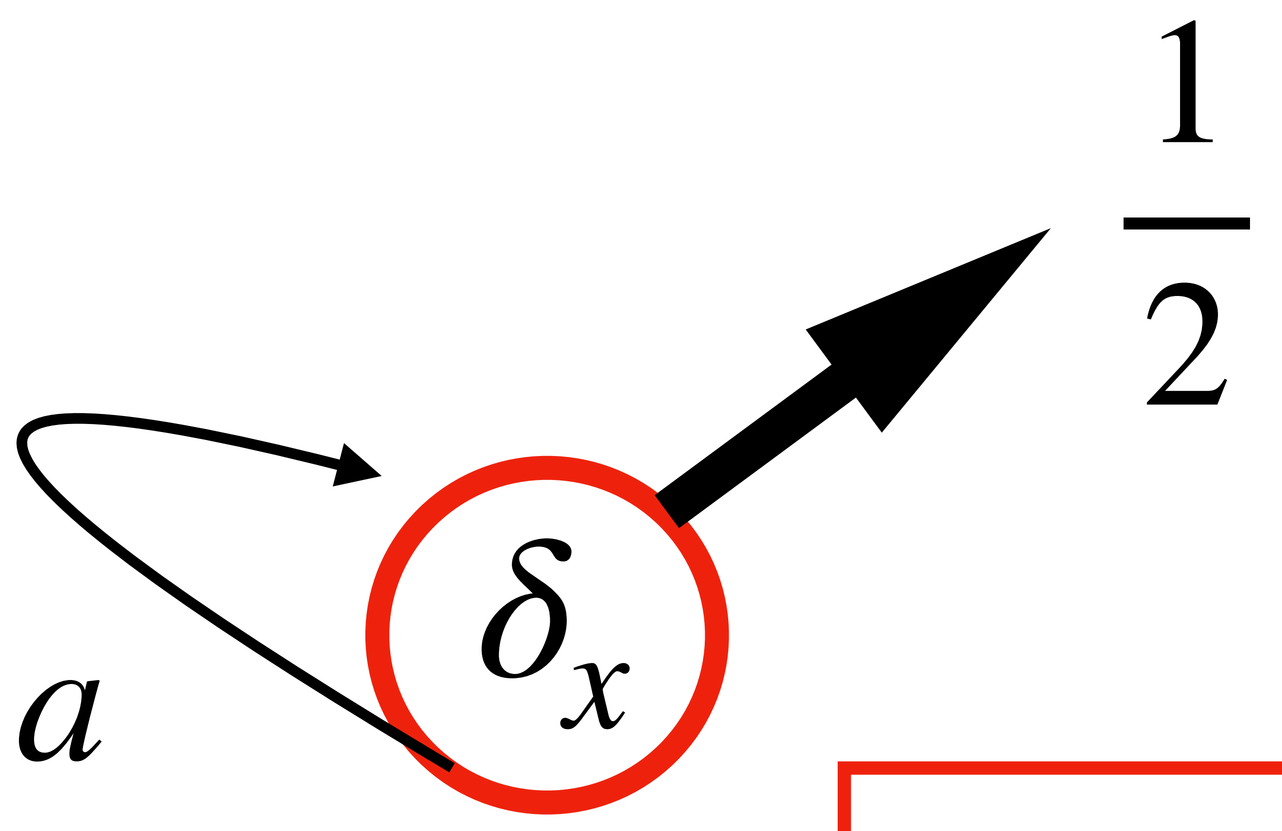
Closure under
algebraic structure

$$\frac{\mathsf{beh}(u(d)) \leq d}{\mu\mathsf{beh} \leq d}$$



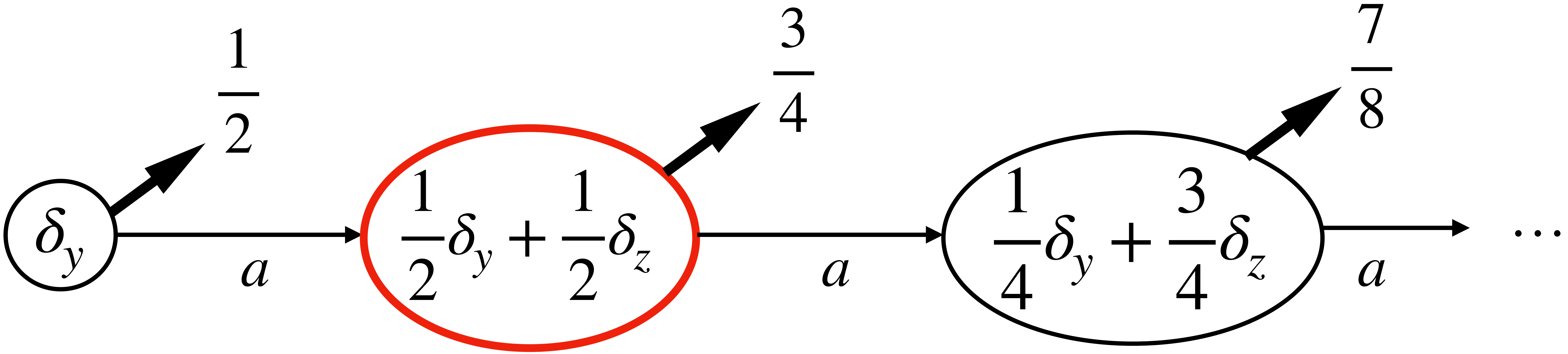
$$\mathsf{beh}(u(d))(\delta_x, \delta_y) = \max\{\frac{1}{2} - \frac{1}{2}, d(\delta_x, \frac{1}{2}\delta_y + \frac{1}{2}\delta_z)\} \leq \frac{1}{2}$$





Now it works!

$$\mathsf{beh}(u(d))(\delta_x, \delta_y) = \max\{\frac{1}{2} - \frac{1}{2}, d(\delta_x, \frac{1}{2}\delta_y + \frac{1}{2}\delta_z)\} \leq \frac{1}{2}$$



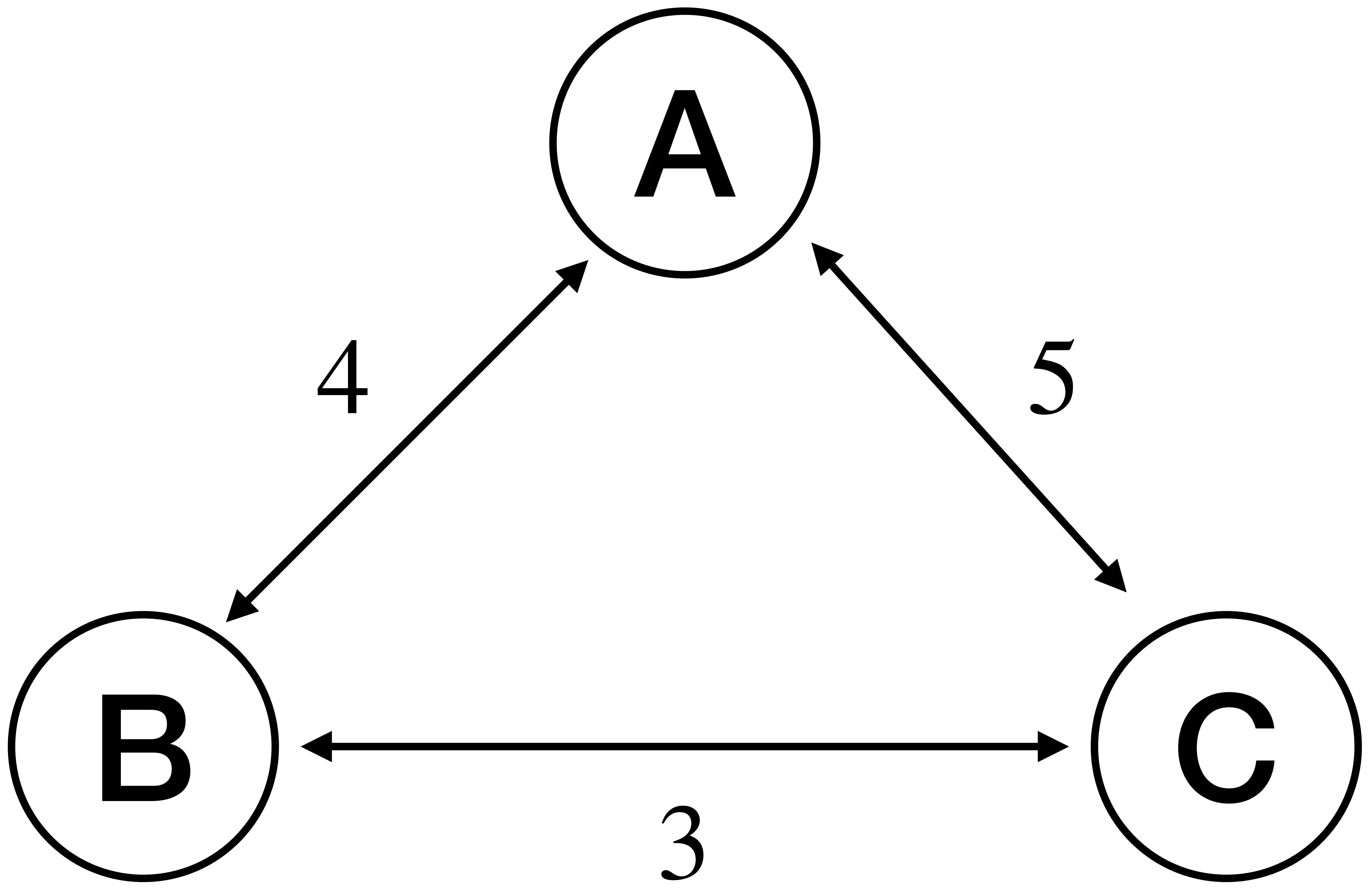
If we can lift $\lambda : TF \Rightarrow FT$ into $\lambda : \overline{T}\overline{F} \Rightarrow \overline{F}\overline{T}$,

then the up-to technique is **sound**

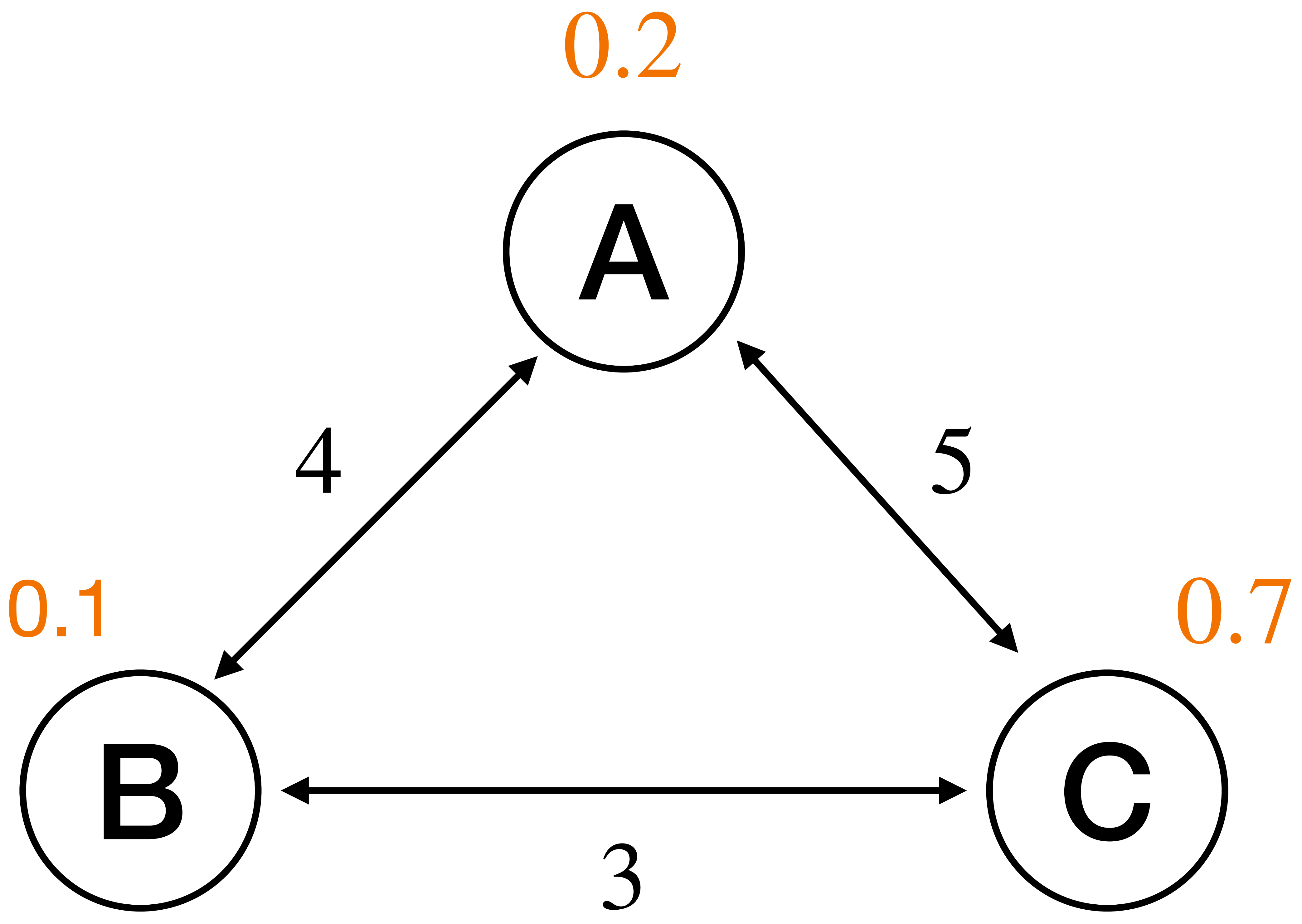
Filippo Bonchi, Petrişan, D., Pous, D. and Rot, J. (2016). A general account of coinduction up-to

Kantorovich lifting

Going from distances on X to $\mathscr{D}X$

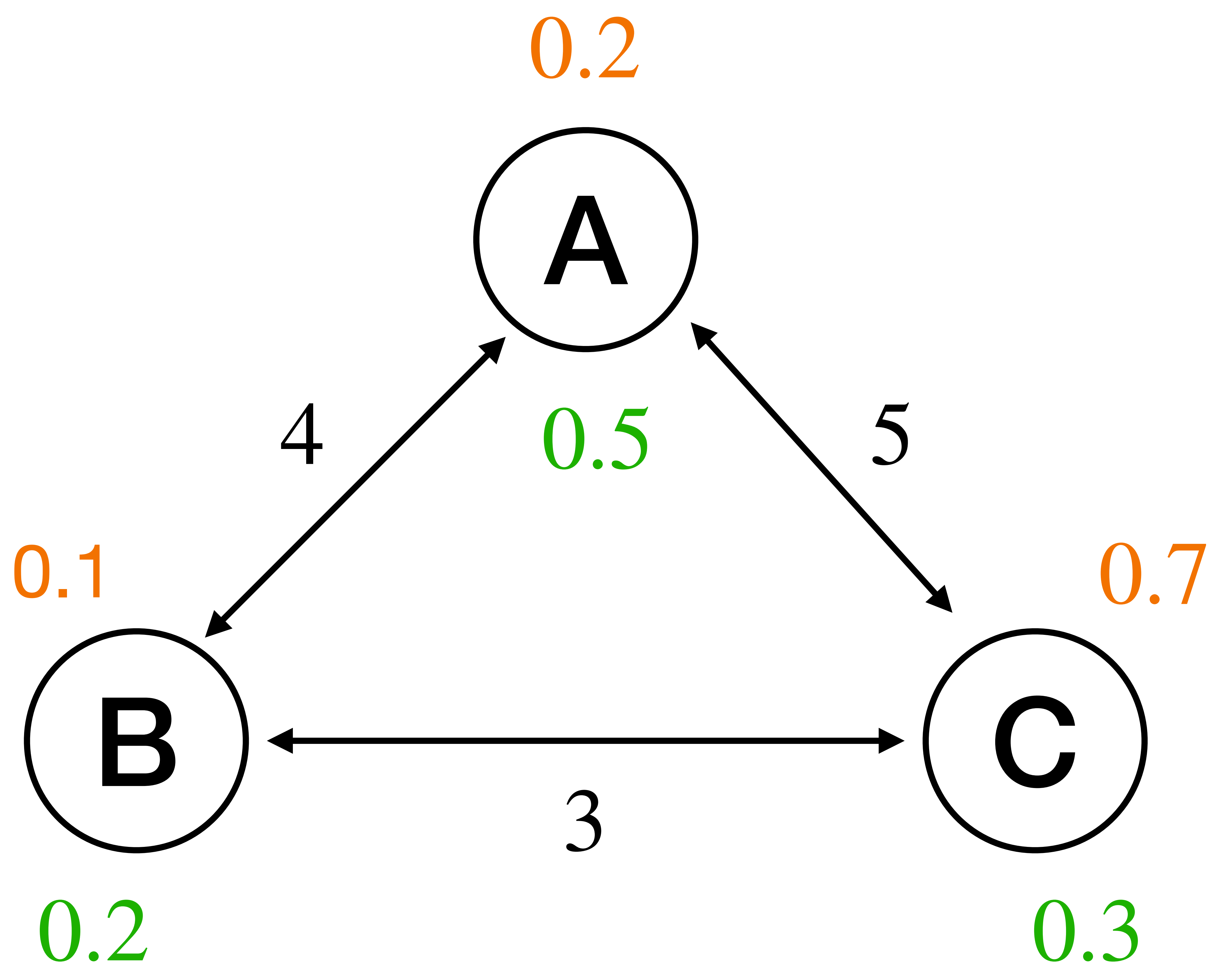


Going from distances on X to $\mathscr{D}X$



Supply

Going from distances on X to $\mathscr{D}X$



Supply

Demand

$$[0,1]^{X\times X} \xrightarrow{\gamma_X} \mathscr{P}([0,1]^X) \xrightarrow{\mathscr{P}(\mathbb{E}_X)} \mathscr{P}([0,1]^{\mathscr{D}X}) \xrightarrow{\alpha_X} [0,1]^{\mathscr{D}X\times\mathscr{D}X}$$

$$[0,1]^{X\times X} \xrightarrow{\gamma_X} \mathscr{P}([0,1]^X) \xrightarrow{\mathscr{P}(\mathbb{E}_X)} \mathscr{P}([0,1]^{\mathscr{D}X}) \xrightarrow{\alpha_X} [0,1]^{\mathscr{D}X\times\mathscr{D}X}$$

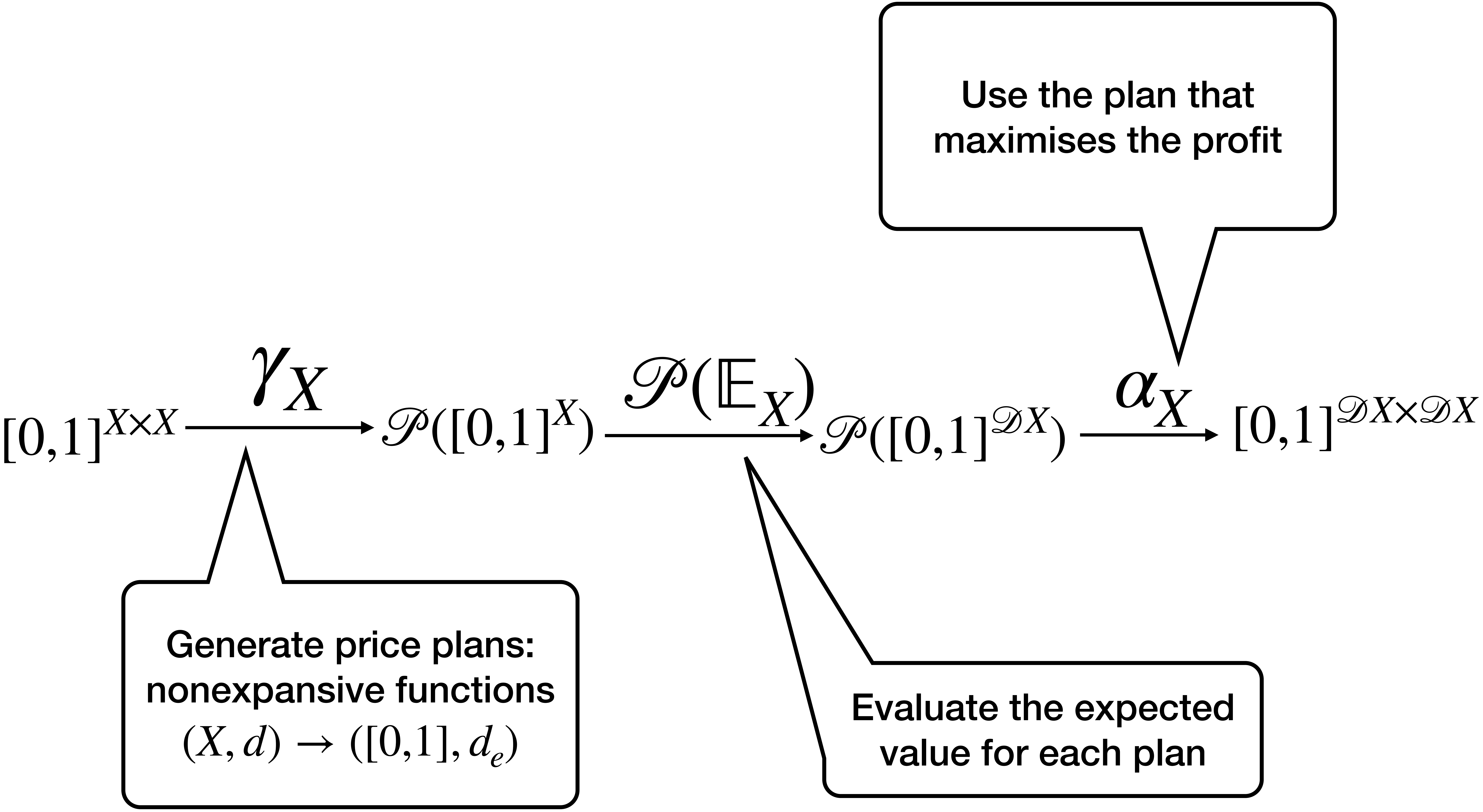
Generate price plans: nonexpansive functions

$(X, d) \to ([0,1], d_e)$

$$[0,1]^{X \times X} \xrightarrow{\gamma_X} \mathscr{P}([0,1]^X) \xrightarrow{\mathscr{P}(\mathbb{E}_X)} \mathscr{P}([0,1]^{\mathscr{D}X}) \xrightarrow{\alpha_X} [0,1]^{\mathscr{D}X \times \mathscr{D}X}$$

Generate price plans:
nonexpansive functions
$(X, d) \to ([0,1], d_e)$

Evaluate the expected
value for each plan


Use the plan that maximises the profit
$[0,1]^{X\times X} \xrightarrow{\gamma_X} \mathscr{P}([0,1]^X) \xrightarrow{\mathscr{P}(\mathbb{E}_X)} \mathscr{P}([0,1]^{\mathscr{D}X}) \xrightarrow{\alpha_X} [0,1]^{\mathscr{D}X\times\mathscr{D}X}$
Generate price plans: nonexpansive functions $(X, d) \to ([0,1], d_e)$
Evaluate the expected value for each plan

$$[0,1]^X \xrightarrow{\mathbb{E}_X} [0,1]^{\mathscr{D}X}$$

$$[0,1]^X \xrightarrow{\mathsf{ev}_F} [0,1]^{FX}$$

Goncharov, S., Hofmann, S., Nora, P., Schröder, L., Wild, P. (2023) “Kantorovich Functors and Characteristic Logics for Behavioural Distances”

Beohar, H., Gurke, S., König, B., Messing, K., Forster, J., Schröder, L. and Wild, P. (2024). Expressive Quantale-Valued Logics for Coalgebras: An Adjunction-Based Approach.

$$[0,1]^X \xrightarrow{\mathsf{ev}_F} [0,1]^{FX}$$

Test one-step behaviour

Goncharov, S., Hofmann, S., Nora, P., Schröder, L., Wild, P. (2023) “Kantorovich Functors and Characteristic Logics for Behavioural Distances”

Beohar, H., Gurke, S., König, B., Messing, K., Forster, J., Schröder, L. and Wild, P. (2024). Expressive Quantale-Valued Logics for Coalgebras: An Adjunction-Based Approach.

$$[0,1]^X \xrightarrow{\mathsf{ev}_F} [0,1]^{FX}$$

Test one-step behaviour

$$\Downarrow$$

$$[0,1]^X \xrightarrow{\Lambda^F_X} \mathscr{P}([0,1]^{FX})$$

We can have multiple evaluation maps

Goncharov, S., Hofmann, S., Nora, P., Schröder, L., Wild, P. (2023) “Kantorovich Functors and Characteristic Logics for Behavioural Distances”

Beohar, H., Gurke, S., König, B., Messing, K., Forster, J., Schröder, L. and Wild, P. (2024). Expressive Quantale-Valued Logics for Coalgebras: An Adjunction-Based Approach.

$$d: X \times X \to [0,1]$$

$$0 \leq d(x, x)$$

$$d(x, y) = d(y, x)$$

$$d(x, z) \leq d(x, y) + d(y, z)$$

Complete lattice

$$(\mathscr{V}, \sqcup)$$

Monoid

$$(\mathscr{V}, \otimes, k)$$

Complete lattice

$(\mathscr{V}, \sqcup)$

Monoid

$(\mathscr{V}, \bigotimes, k)$

$([0,1], \oplus)$

Complete lattice

$$(\mathscr{V}, \sqcup)$$

Monoid

$$(\mathscr{V}, \otimes, k)$$

$$([0,1], \oplus)$$

$$(\{0,1\}, \wedge)$$

Complete lattice

$(\mathscr{V}, \sqcup)$

Monoid

$(\mathscr{V}, \otimes, k)$

$([0,1], \oplus)$ $(\{0,1\}, \wedge)$ $([0,\infty], \max)$

$d\colon X \times X \to \mathscr{V}$

$$
\begin{aligned}
k &\sqsupseteq d(x, x) \\
d(x, y) &= d(y, x) \\
d(x, z) &\sqsupseteq d(x, y) \otimes d(y, z)
\end{aligned}
$$

$$d\colon X \times X \to \mathscr{V}$$

$$k \sqsupseteq d(x, x)$$

$$\sout{d(x, y) = d(y, x)}$$

$$d(x, z) \sqsupseteq d(x, y) \otimes d(y, z)$$

$$d \colon X \times X \to \mathscr{V}$$

$$k \sqsupseteq d(x, x)$$

$$\sout{d(x, y) = d(y, x)}$$

$$d(x, z) \sqsupseteq d(x, y) \otimes d(y, z)$$

Directed metrics

If we can lift $\lambda : TF \Rightarrow FT$ into $\lambda : \overline{T}\overline{F} \Rightarrow \overline{F}\overline{T}$,

then up-to technique is **sound**

If we can lift $\lambda : TF \Rightarrow FT$ into $\lambda : \overline{T}\overline{F} \Rightarrow \overline{F}\overline{T}$,

then up-to technique is **sound**

To get soundness it enough to show that:

1) $\overline{F}\ \overline{T} = \overline{FT}$

2) Predicate liftings interact in the right way with λ

If we can lift $\lambda : TF \Rightarrow FT$ into $\lambda : \overline{T}\overline{F} \Rightarrow \overline{F}\overline{T}$,

then up-to technique is **sound**

To get soundness it enough to show that:

1) $\overline{F}\ \overline{T} = \overline{FT}$

2) Predicate liftings interact in the right way with λ

For Kantorovich, we know that $\overline{T}\ \overline{F} \sqsubseteq \overline{TF}$ for free

Compositionality

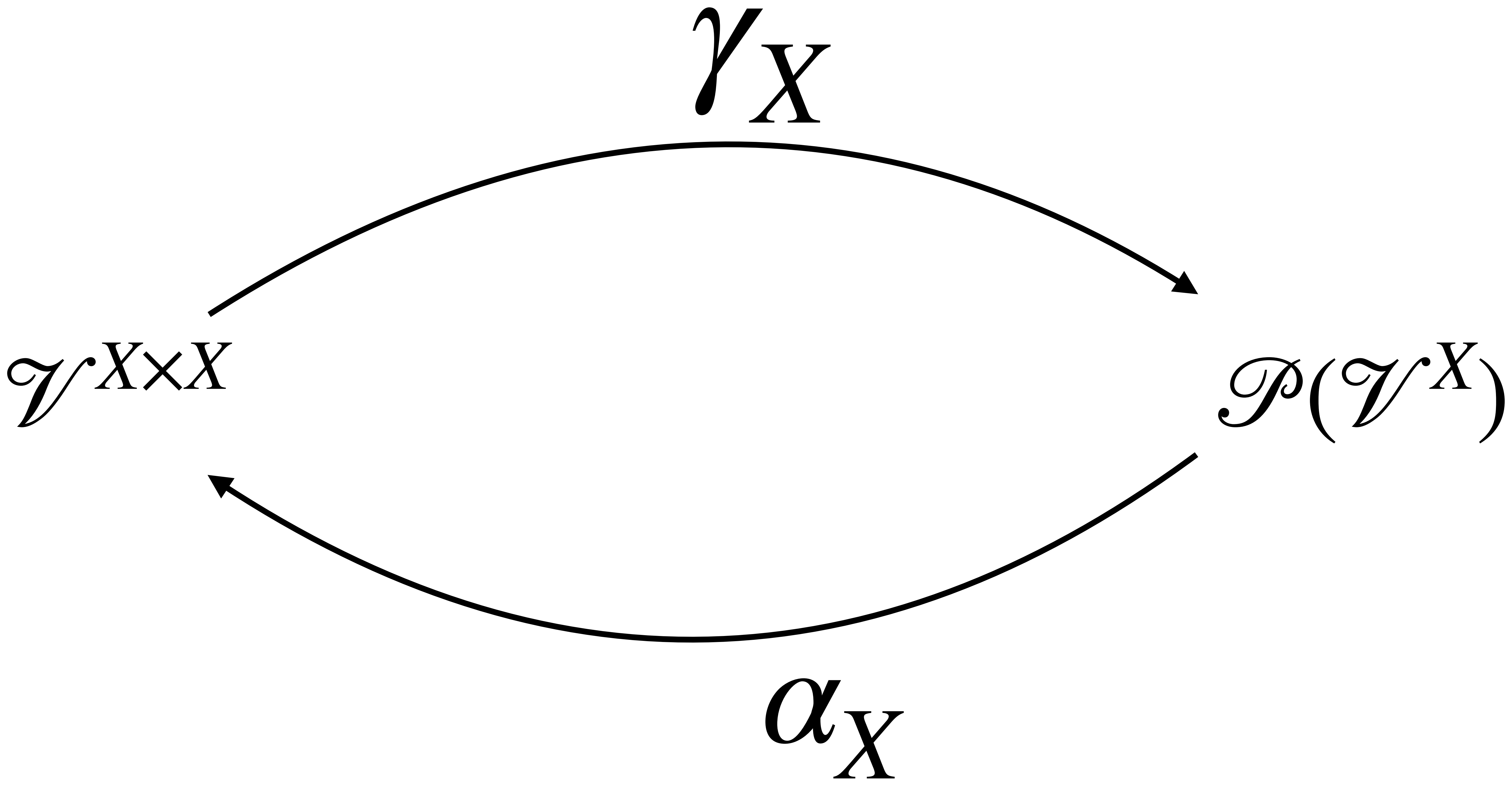
$$\overline{F}\ \overline{T} = \overline{FT}$$

Sound up-to technique for FT-coalgebras

$$F = C_B \mid \mathsf{Id} \mid \prod_{i \in I} F_i \mid F_1 + F_2$$

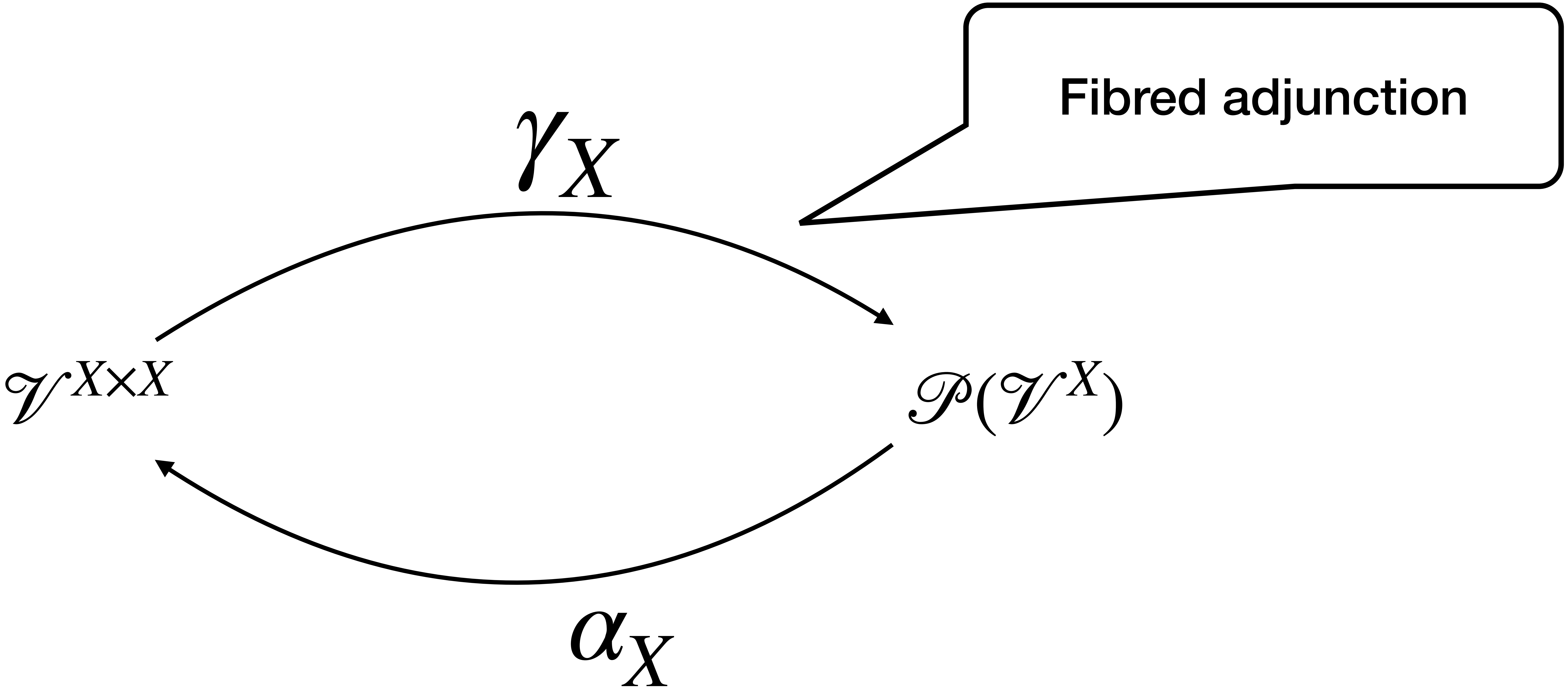
To make that work, we construct appropriate sets of evaluation maps that interact well with the distributive law

Core results of the result come from fibred category theory



Filippo Bonchi, König, B. and Petrişan, D. (2023). Up-to techniques for behavioural metrics via fibrations.

Core results of the result come from fibred category theory



Filippo Bonchi, König, B. and Petrişan, D. (2023). Up-to techniques for behavioural metrics via fibrations.

Quantale-valued behavioural distances for a wide class of transition systems

Quantale-valued behavioural distances for a wide class of transition systems

Sound and succinct way of proving bounds on the distances

Quantale-valued behavioural distances for a wide class of transition systems

Sound and succinct way of proving bounds on the distances

Kantorovich lifting is flexible and also allowed us to tackle coproducts