Processes Parametrised by an Algebraic Theory

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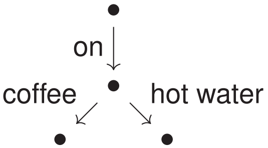
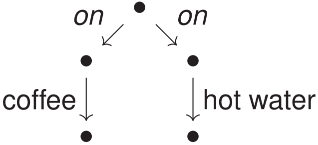
An evil coffee machine from our basement

How it should be



In our office



Our coffee machine is trace equivalent to the good one, but not bisimilar

Milner's Algebra of Regular Behaviours

Syntax:

$$e, e_i ::= 0 \mid v \mid e_1 + e_2 \mid ae \mid \mu v\ e$$

Operational model:

$$Q \to \mathcal{P}_\omega(V + A \times Q)$$

Denotational model: Behaviours - Equivalence classes of bisimilar trees

Star fragment:

$$1 \mapsto \underline{u} \qquad a \mapsto a\underline{u} \qquad e_1 + e_2 \mapsto e_1 + e_2 \qquad e_1 e_2 \mapsto e_1[e_2/\underline{u}] \qquad e^* \mapsto \mu v\ (e[v/\underline{u}] + \underline{u})$$

GKAT

Syntax:

$$e, e_i ::= 0 \mid 1 \mid b \in \mathtt{At} \mid a \in \Sigma \mid e_1 +_b e_2 \mid e_1 e_2 \mid e^{(b)}$$

Operational model:

$$Q \to (2 + A \times Q)^{\mathtt{At}}$$

Recent work by Schmid, Kappe, Kozen and Silva provided new axiomatisation omitting right cancellation and left distributivity and provided semantics in terms of trees

Why are those two related?

GKAT:

$$Q \to (2 + A \times Q)^{\mathtt{At}} \cong Q \to (1 + (1 + A \times Q))^{\mathtt{At}}$$

Star fragment of ARB:

$$Q \to \mathcal{P}_\omega(1 + A \times Q)$$

Both $\mathcal{P}_\omega$ and $(1 + \mathrm{Id})^{\mathtt{At}}$ are **monads**.

Our generic framework

Syntax:

$$e, e_i ::= 0 \mid v \mid \sigma(e_1, \ldots, e_n) \mid ae \mid \mu v\ e$$

Semantics:

$$\epsilon(v) = \eta^M(v) \qquad \epsilon(\sigma(e_1, \ldots, e_n)) = \sigma(\epsilon(e_1), \ldots, \epsilon(e_n))$$

$$\epsilon(ae) = \eta^M((a, e)) \qquad \epsilon(\mu v\ e) = \epsilon(e)[\mu v\ e // v]$$

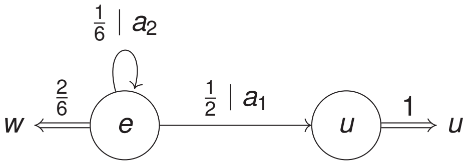
Operational model: coalgebras for the endofunctor

$$B_M = M(V + A \times \mathrm{Id})$$

where M is a monad presenting algebraic theory (S, E), which has designated constant element and operations of finite arity

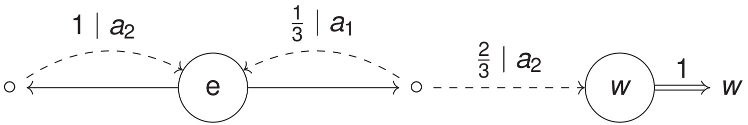
Example 1 - distribution monad

$e = \mu v\ (a_1 u +_{\frac{1}{2}} (a_2 v +_{\frac{1}{3}} w))$ is



Example 2 - convex powerset monad

$$e = \mu v \left(\left(a_1 v +_{\frac{1}{3}} a_2 w \right) + a_2 v \right)$$



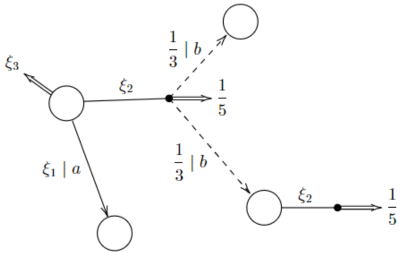
Probabilistic GKAT?

Syntax:

$$e, e_i ::= 0 \mid 1 \mid b \mid p \in P \mid e_1 +_p e_2 \mid e_1 ?_b e_2 \mid e_1 ; e_2 \mid e^{(b?)} \mid e^{(p)}$$

Operational model:

$$Q \to \mathcal{D}_\omega(2 + A \times Q)^{\mathtt{At}}$$



Denotational model:

- Partial functions: $\mathtt{At}^+ \to \mathcal{D}_\omega(2 + A)$
- Probabilistic Guarded Languages: $(\mathtt{At} \cdot [0,1] \cdot P)^* \cdot \mathtt{At} \cdot [0,1] \cup (\mathtt{At} \cdot [0,1] \cdot P)^\omega$

Questions?